PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Phil. Trans. R. Soc. Lond. A 1946 **240**, 1-57
doi: 10.1098/rsta.1946.0002

THE CALCULATION OF VARIABLE HEAT FLOW IN SOLIDS

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(*Received* 22 *November* 1944)

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INTRODUCTION

1. GENERAL SURVEY

In many contexts, in laboratory work, in industrial processes and in engineering practice, situations arise in which it is required to determine the flow of heat in bodies under non-steady conditions. In some cases this can be done by direct experiment, and a few cases are sufficiently simple for formal analytical solutions of the appropriate equations to be obtained and evaluated. But direct experiment is often difficult or impossible, especially under practical conditions of manufacture or operation; and often the conditions of the problem, such as inhomogeneity or the shape of the material, or variation of its thermal properties with temperature, either make the formal solution so complicated that its numerical evaluation is impracticable, or put a formal solution out of the question altogether.

[Published 2 August 1946

The primary purpose of the work which forms the subject of this paper was the development of a practical method of evaluating solutions of the equations of heat conduction in solids, under non-steady conditions, which should be free of the restriction to constancy of thermal properties. But the method developed clearly has wider applications, for example, to the formally analogous equation of diffusion when the diffusion coefficient depends on the concentration of the diffusing substance. And, although the equations are considerably different in form, it seems that in principle the method could be applied to the equations of the laminar boundary layer in the motion of a viscous fluid. But so far the only applications of the method which have actually been made are to the equation of heat conduction, and the argument will be presented in terms appropriate to these applications.

Formally, the calculation of variable heat flow in solids is concerned with the evaluation of solutions of the equation

$$\frac{\partial\theta}{\partial t} = D(\theta)\,\nabla^2\theta, \tag{1·1}$$

where θ is the temperature* and, for a material of thermal conductivity K, density ρ and specific heat σ, the diffusivity D is given by

$$D = K/\rho\sigma \tag{1·2}$$

and may depend both on position (in heterogeneous materials) and on temperature. Since one of the important features of the present treatment is that it is not restricted to materials whose thermal properties are independent of temperature, this will be emphasized by writing D explicitly as a function of temperature as in (1·1); the variation of D with position, in heterogeneous materials, will be taken as understood.

The feature which characterizes the problem from the point of view of the present method of evaluating solutions of this equation (1·1) is not so much the form of the equation itself as the form of the boundary of the domain over which solutions of the equation are required, the solutions having to satisfy given conditions on that boundary. These conditions are normally given over a closed boundary in the relevant space co-ordinates, and over a single boundary in the time co-ordinate; the domain of integration is open in the time direction, in that the solution has not to satisfy any conditions, other than the space-boundary conditions, at a later time. It is this feature which makes the present method applicable. A formally similar situation occurs in the case of the equations of the laminar boundary layer, in which the field of integration is open in one co-ordinate though in this case a spatial one. On the other hand, when the domain of integration has a closed boundary, as in the calculation of *steady* states in the case of heat conduction, the method is inapplicable. This difference in kind between the problems of calculating steady and variable heat flow or diffusion must be clearly recognized in any consideration of possible applications of the present method.

Among the contexts in which information regarding heat conduction under non-steady conditions is of great importance is the industry concerned with the manufacture of steel. The thermal properties of steel vary considerably with temperature over the relevant range, and no treatment of the heat flow in steel which does not endeavour to take account of this, at least approximately, can be regarded as satisfactory for application to the practical

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* When the conductivity depends on the temperature, θ in this equation should be replaced by a modified temperature (Θ) as defined in § 8.

problems of the industry. An extensive series of accurate measurements of the thermal properties of a number of typical steels over a wide range of temperatures has recently been made at the National Physical Laboratory (Physics Department, N.P.L. 1939) at the request of the Thermal Treatment Sub-Committee of the Alloy Steels Research Committee. This showed clearly the need for a method of calculation of heat flow in substances whose thermal properties vary with temperature, and the work with which this paper is concerned was carried out under the auspices of the Thermal Treatment Sub-Committee of the Alloy Steels Research Committee, with direct reference to the practical problems of calculating the temperature distribution, and its variation with time, in steel ingots and plates in various heat treatments.

In this context, as in many others, many problems of practical importance are concerned with processes in which the temperature distribution is changing with time, and then the determination of the temperature distribution is not easy. In the past, various methods of attack have been tried, and these can for convenience be divided into four categories.

(i) *Experimental determination.* There are a few cases mentioned in the literature of attempts to measure the temperature distribution in large masses of steel during heating. The technique, however, is difficult, expensive, and the results apply only to the particular cases examined. Such experimental data is very limited in extent, and its accuracy may be open to question. The present work has emphasized the difficulties in such experimental work, and the caution with which results must be regarded unless checked by at least a rough calculation.

(ii) *Formal analytical solution.* For constant diffusivity and suitable initial and terminal conditions, analytical solutions can be obtained, though the resulting formulae may be inconvenient for use in finding actual numerical values. But for a material in which the diffusivity varies with temperature and, moreover, for which this variation is given by an empirical curve as it is in practice and not by an analytical formula, there is little prospect of obtaining a formal solution except as an approximation or as a special case (some examples of approximate formal solutions will be given later in § 13).

Hence, though many attempts have been made to apply the formal solutions of equation (1·1) to practical problems in industry, simplifying assumptions have usually had to be made, which may be so drastic that the results are often of little immediate practical value, and in certain circumstances may actually be misleading. These assumptions usually include constant thermal properties of the material, and often specially simple surface conditions, such as the surface temperature rising or falling instantaneously, or at a steady rate. Further, the treatment of the case of two dissimilar materials in contact leads to very heavy formulae, and numerical evaluation of the resulting expressions would be so laborious as to be almost prohibitive.

(iii) *Graphical treatment.* Schmidt (1924,* 1937) has given an approximate graphical method of evaluating solutions of the equation of heat conduction in one dimension, using finite differences both in space and in time; this has been extended by Nessi & Nisolle (1928) to radial heat flow in a circular cylinder. The approximation is based on a graphical discussion,

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* Despite many enquiries, we have been unable to trace in this country a copy of the volume in which Schmidt's original paper was published. Our information regarding his method is derived from references to it in other works, such as Trinks (1926), Appendix, p. 323.

and no thorough investigation is made of the errors involved by the use of finite differences. For the integration with respect to time, the crude integration formula

$$\int_{t_0}^{t_0+\delta t} f(t)\,dt = f(t_0)\,\delta t$$

is used. For a material of constant diffusivity, the work is much simplified by taking time intervals related to the space intervals in a way depending on the diffusivity. The technique of calculation is difficult and laborious to apply unless constant thermal properties are assumed; further, in the case of steel and other materials of high diffusivity it becomes exceedingly tedious and difficult to apply with any accuracy.

An equivalent numerical process for constant diffusivity, probably less laborious to apply than Schmidt's graphical process and more accurate in its results, has been given by McAdams (1942).

(iv) *Methods involving the use of models.* Such methods attempt to simulate the heat flow by electrical or hydraulic analogies (see, for example, Paschkis & Baker (1941, 1942) and Moore (1936)). At first sight, it might appear that use of such models would provide an easy solution to this difficult problem, but actually it is found that the practical difficulties of construction and calibration are considerable. Further, without considerable elaboration, they cannot be applied to heat conduction in a material whose diffusivity varies continuously with the temperature.*

The work here described was concerned with the development and application of a method of calculating variable heat flow which would be free from the restrictions to constant diffusivity, or to particular time variations of surface temperature or surface heat transfer, or to a particular form of relation between these quantities, and also would be quicker and easier to carry out than Schmidt's graphical method. Such a development, it was considered, would represent a considerable advance owing to the much wider scope of a method free from these restrictions. Further, the closer approximation to practical conditions which its use would permit would enable results of acceptable accuracy to be obtained over a much wider range of problems of practical significance.

The method developed depends on the use of an approximation by which the partial differential equation (1·1) is replaced by a set of simultaneous ordinary differential equations, which are of a form very suitable for mechanical solution by the differential analyser (Bush 1931; see also Hartree 1938). This approximation is formally the same as one of those made in Schmidt's method (see (iii) above), but the present treatment of the resulting set of ordinary differential equations is quite different from that of Schmidt, and avoids the main restrictions on the practical application of Schmidt's method. Further, in the present treatment no further approximation need be made, except in the treatment of a transformation occurring at a fixed temperature with emission or absorption of latent heat, though the work can be simplified by the use of further approximations if the accuracy obtained by use of

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* In one hydraulic instrument (Moore 1936), provision is made for treating variable diffusivity by an approximation which involves regarding the diffusivity as constant and uniform over successive periods of time between which it changes discontinuously. This is a better approximation than taking the diffusivity as a constant, but it seems very artificial, and a more realistic treatment seems desirable.

them can be shown to be adequate; whereas Schmidt's method essentially involves the use of a further approximation and, moreover, a crude one as already mentioned. In the present work there is no difficulty in taking into account continuously the variation of the thermal properties with temperature, including the heat of transformation at a change point, and, in the case of an ingot cooling in a mould, the radiative transfer of heat through the gap between the outer surface of the ingot and the inner surface of the mould. Also it is possible to extend the method to one-dimensional heat conduction in a slab consisting of layers of different materials in contact over a plane normal to the flow of heat, and to the corresponding axially symmetrical cylindrical case.

Also, as the work proceeded, it was found practicable (still without restriction to constant diffusivity) to treat heat conduction in a square prism and two-dimensional heat conduction in a circular cylinder without the restriction to axial symmetry, as well as one-dimensional heat flow in a slab and radial heat flow in a cylinder. Further, the use of the differential analyser provides a much quicker, less laborious and more accurate technique for evaluating solutions than Schmidt's graphical method, so that the much greater range of problems which can now be handled is accompanied by a quicker and easier process of handling them.

2. Scope of the present paper

The contents of this paper can conveniently be divided into four parts. The first is concerned with the basic approximation by which the partial differential equation is reduced to a set of ordinary differential equations, and with the results of using this approximation in the various forms of the equation of heat conduction appropriate to bodies of different shapes and with different degrees of symmetry in the heat flow. The second part is concerned with some results of calculations of variable heat flow in materials for which the thermal properties are independent of temperature. The third deals with the general case of heat conduction in a material whose thermal properties vary with temperature. The fourth deals with the treatment of an ingot cooling in a mould, when heat flow is by conduction through the material of the ingot and the mould, and by radiation across the gap between them.

Although, as already emphasized, the main purpose of the work was the development of a method of calculating variable heat flow free from the restriction to constant diffusivity, there are several reasons why the work considered in the second part was undertaken and why the results are important to the work as a whole. First, when the thermal properties are constant, it is possible to find cases of variable heat flow in which both the partial differential equation (1·1) and the set of ordinary differential equations by which it is replaced have formal solutions. These can then be taken as test cases to examine both the magnitude of the errors introduced by the use of the analytical approximation on which the method is based and also the accuracy of the differential analyser in the solution of the set of equations arising from the use of this approximation. Further, from the point of view of the technique of the use of the differential analyser in work of this kind, which was rather different in character from anything previously undertaken, it seemed best to start on the simpler case of constant diffusivity for exploratory work and then, if this proved satisfactory, to introduce the variation of diffusivity with temperature.

Before a solution of the equation of heat conduction can be evaluated, the conditions at the surface of the material must be specified. These may be of three kinds. Either the surface temperature or the surface heat transfer may be given as functions of the time, or the relation between the surface heat transfer and the surface temperature may be given (and may or may not depend on the time). The simplest surface condition to deal with by the present method is a given variation of surface temperature with time. But the treatment of the heat flow *within* the material is independent of the surface condition, and the method can be extended to cover the other kinds of surface conditions mentioned.

However, in the application of the method to the evaluation of the temperature distribution inside steel masses in furnaces, the quantitative laws governing the flow of heat from the furnace into the steel are not known with sufficient precision, nor are the conditions in a furnace sufficiently well defined to be used as the basis for quantitative calculations of the internal temperature distribution. On the other hand, under experimental conditions, and sometimes, at least at high temperatures, under practical works conditions, the surface temperature can be determined. Consequently attention was first concentrated on the case, which happens also to be the simplest to handle, in which the surface temperature is given as a function of the time.

The treatment is here simplified by restricting it to solids of certain simple shapes, the slab, cylinder and square prism. This is not a limitation of the method, but the many practical cases can be covered with sufficient accuracy by the results for these cases. Further, in order that the methods should be of any value in application to less simple geometrical shapes, it would be necessary to know the temperature distribution all over the surface and this information is usually not available.

On the general question of the accuracy required in this work, it should be realized that the thermal properties of steel are not reproducible to an accuracy which would justify any great refinements in methods of solving the heat-flow equations for practical purposes. Even if values in the transformation range (say 700–750° C) are omitted, the specific heats of different specimens of nominally the same steel, at the same temperature, differ by 2–5 % (cf. steels 5 and 5*, *Second Report of the Alloy Steels Research Committee* (1939), Table 47; above 900° the difference is 8–15 %). So approximate methods of evaluating solutions of the heat-flow equations which are correct to 1 or 2 % as compared with accurate solutions of those equations are to be regarded as adequate for practical purposes.

PART I. THE REDUCTION OF THE PARTIAL DIFFERENTIAL EQUATION OF HEAT CONDUCTION TO A SET OF ORDINARY DIFFERENTIAL EQUATIONS

3. Heat conduction in the interior of the conducting material

As already mentioned, the treatment of heat conduction within the conducting material is independent of the surface conditions, and this treatment is the main subject of this section. In so far as reference is made to surface conditions, it will be supposed that these are in the form of given time variations of surface temperatures. The treatment of other surface conditions is considered in § 4.

The heat-conduction equation is a partial differential equation, and the evaluation of solutions of partial differential equations as such, either mechanically or numerically, is hardly practicable except in the few cases for which there is available an analytical solution which can be evaluated numerically. Otherwise, it is necessary to replace the partial differential equation by an approximate ordinary differential equation or set of such equations before it can be handled either mechanically or numerically. When such an approximate solution has been evaluated, it is often possible to estimate the error introduced and, if necessary, to apply an approximate correction for it. An example will be found later in the present work (see § 14 (ii)).

In the case of the equation of heat conduction in one dimension, there are two alternative methods of making such an approximation. One method is to replace the *time* rate of change of temperature by the temperature change in a finite time interval; the accuracy of such a method has been examined (Hartree & Womersley 1937) and it has been applied successfully to the transient temperature behaviour in a dielectric in an alternating field, when the rate of generation of heat increases exponentially with the temperature (Copple, Hartree, Porter & Tyson 1939). Unless the equation is linear (and, in practice, often even when it is linear), boundary conditions have to be fitted by running a number of trial solutions from one end of the space range until one is found which satisfies the condition at the other end (see Copple *et al.* 1939, Appendix II); this makes the method rather lengthy.

Alternatively, the *space* derivative may be replaced by a corresponding finite difference; the treatment of boundary conditions is simpler in this method and it is practicable with simpler equations. The latter method was used in the work with which this report is concerned.

The method will be presented here as applied to equation (1·1); its extension to heat conduction in a material whose conductivity varies with the temperature will be considered in § 9.

(*a*) *One-dimensional heat flow.* The simplest case is that of one-dimensional heat flow. Consider an infinite plane slab, of thickness $2l$, heated uniformly over each face. Take the x-axis normal to the plane of the slab, with origin at the mid-plane, and consider the slab divided into $2n$ layers, each of thickness $\delta x = l/n$, by parallel planes $x = x_j = j\,\delta x$ with j integral and $-n \leqslant j \leqslant n$ (the values $j = \pm n$ give the surfaces of the slab). Let θ_j be the temperature on the plane x_j.

Then it is easily shown from the Taylor expansions of $\theta_{j\pm 1}$ in terms of θ and its derivatives at x_j, that

$$\theta_{j+1} - 2\theta_j + \theta_{j-1} = (\delta x)^2\,(\partial^2\theta/\partial x^2)_j + \tfrac{1}{12}(\delta x)^4\,(\partial^4\theta/\partial x^4)_j + O\,(\delta x)^6 \qquad (3\cdot 1)$$

Hence an approximation to $(\partial^2\theta/\partial x^2)_j$ neglecting second-order terms in $(\delta x)^2$ is

$$\left(\frac{\partial^2\theta}{\partial x^2}\right)_j = (\theta_{j+1} - 2\theta_j + \theta_{j-1})/(\delta x)^2. \qquad (3\cdot 2)$$

A corresponding approximation to the temperature gradient $(\partial\theta/\partial x)_j$ will be required later to the same order of approximation; this is given by

$$\left(\frac{\partial\theta}{\partial x}\right)_j = \frac{\theta_{j+1} - \theta_{j-1}}{2(\delta x)}. \qquad (3\cdot 3)$$

The essential approximation of the present work lies in using these approximations to the space derivatives in the heat-flow equation (1·1).

It can be seen from (3·1) that this is equivalent to replacing the actual temperature distribution between x_{j-1} and x_{j+1} by a cubic. It can also be seen from (3·1) that the errors introduced by this approximation can be decreased indefinitely by decreasing (δx), that is, by increasing the number of intervals into which the thickness of the slab is divided. Use of a large number n of intervals has, however, practical disadvantages, and the time required to obtain a solution on the differential analyser (or, probably, by any alternative means) increases at least as rapidly as n^2. Hence it is important to determine the smallest value of n required to give adequate accuracy for practical purposes. This was one of the main objects of the work considered in Part II.

If the diffusivity varies with the temperature, let $D(\theta)$ be the diffusivity at temperature θ. Then for the time rate of change of temperature at $x = x_j$, (1·1) becomes

$$\frac{\partial \theta_j}{\partial t} = D(\theta_j)\left(\frac{\partial^2 \theta}{\partial x^2}\right)_j,$$

and use of the approximation (3·2) gives

$$\frac{\partial \theta_j}{\partial t} = D(\theta_j)\,\frac{\theta_{j+1} - 2\theta_j + \theta_{j-1}}{(\delta x)^2}. \tag{3·4}$$

There are $2n-1$ such equations for the time variations of the temperatures θ_j from $j = -(n-1)$ to $j = +(n-1)$; θ_n and θ_{-n} are the surface temperatures and are to be regarded for the present as given, as already explained.

If the heating is the same on the two faces of the slab, the internal temperature distribution will be symmetrical, so that $\theta_j = \theta_{-j}$ and in particular $\theta_1 = \theta_{-1}$, so the equation (3·4) for $j = 0$, that is, for the temperature on the central plane, becomes

$$\frac{\partial \theta_0}{\partial t} = D(\theta_0)\,\frac{2(\theta_1 - \theta_0)}{(\delta x)^2}. \tag{3·5}$$

It is then only necessary to work with equations (3·4) for one-half of the slab thickness (those for positive values of j for convenience), and (3·5) for the central temperature. To the approximation used here, then, the partial differential equation (1·1) is replaced by the set of simultaneous ordinary differential equations (3·4) and (3·5).

(*b*) *Radial heat flow in circular cylinder.* The equation of radial heat flow in a circular cylinder is

$$\frac{\partial \theta}{\partial t} = D(\theta)\left(\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r}\right). \tag{3·6}$$

Let the radius a be divided into n equal intervals $\delta r = a/n$, and let $r_j = j\,\delta r$ be the outer boundary of the jth interval and θ_j the temperature there. Then use of the approximation (3·2 and 3·3) (with r as independent variable) in (3·6) gives

$$\frac{\partial \theta_j}{\partial t} = D(\theta_j)\,\frac{(2j+1)\,\theta_{j+1} - 4j\theta_j + (2j-1)\,\theta_{j-1}}{2j\,(\delta r)^2}. \tag{3·7}$$

This only holds for $j \geqslant 1$; it is indeterminate for $j = 0$; but by symmetry, $\partial\theta/\partial r = 0$ at $r = 0$, so if $\theta = \theta_1$ at $r = \delta r$, $\theta = \theta_0 + (\theta_1 - \theta_0)\, r^2/(\delta r)^2$ approximately, for (δr) sufficiently small and $r < \delta r$. Hence

$$\left(\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}\right)_{r=0} = 4(\theta_1 - \theta_0)/(\delta r)^2,$$

and so (3·6) gives

$$\frac{\partial\theta_0}{\partial t} = D(\theta_0)\,\frac{4(\theta_1 - \theta_0)}{(\delta r)^2}. \tag{3·8}$$

An alternative derivation of this equation will be given later (see subsection (*d*) below). Equation (3·7) for $j = 1$ to $(n-1)$, and equation (3·8) for $j = 0$, are those which have to be solved to give the time variation of temperature distribution in the cylindrical case. A similar argument applies to radial heat conduction in a sphere, but this case is not of practical interest.

The coefficient 4 in (3·8), compared with the coefficient 2 in (3·5) should be noted. It arises from the greater concentration of the heat flow towards the axis of the cylinder, as compared with the concentration towards the central plane in the slab; it might be expected and can be verified, that the still greater concentration towards the centre of a sphere makes this coefficient 6.

An alternative method, for a *hollow* cylinder of internal radius b, is to use

$$\lambda = \log\,(r/b) \tag{3·9}$$

as independent variable. Then (3·6) becomes

$$\frac{\partial\theta}{\partial t} = e^{-2\lambda} D(\theta)\,\frac{\partial^2\theta}{\partial\lambda^2}, \tag{3·10}$$

so the heat flow can be treated as one-dimensional heat flow in an inhomogeneous solid with diffusivity proportional to $e^{-2\lambda}$ (and perhaps varying with temperature also). Equations (3·4) and (3·5) then apply, with this non-uniform diffusivity.

This treatment is clearly inapplicable to a *solid* cylinder, but it might be convenient for a hollow cylinder for which a/b is not a ratio of fairly small integers.

(*c*) *Two-dimensional heat flow in a prism of square section.* For a prism of square section, it is most convenient to use the equation of heat flow in two dimensions in its Cartesian form

$$\frac{\partial\theta}{\partial t} = D\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) \tag{3·11}$$

rather than in its polar form of which (3·6) is the special case for axial symmetry. Let $2a$ be the side of the square section. The centre of a section will be taken as origin and the sides of the square will be regarded as divided into segments of length $\delta x = \delta y = \delta s = a/n$. The temperature at the point $x = j\,\delta s$, $y = k\,\delta s$ will be written $\theta_{j,k}$.

The x and y derivatives in (3·11) can both be replaced by corresponding finite differences (3·2), and (3·11) then becomes

$$\frac{\partial\theta_{j,k}}{\partial t} = D(\theta_{j,k})\,\frac{\theta_{j+1,k} + \theta_{j,k+1} + \theta_{j-1,k} + \theta_{j,k-1} - 4\theta_{j,k}}{(\delta s)^2}. \tag{3·12}$$

The surface temperature need not be uniform, but, if not uniform, its time variation will be taken as given at each surface point $x = j\,\delta s$, $y = \pm a$ and $x = \pm a$, $y = k\,\delta s$.

If there is no symmetry in the temperature distribution, the number of independent equations to be solved is $(2n-1)^2$, but if there is some degree of symmetry in the heating, the number is reduced considerably. In the simplest case of full symmetry (which includes the case of uniform heating) the number is reduced to $\frac{1}{2}n(n+1)$; it is then only necessary to consider the heat flow and temperature distribution in and on the boundary of an octant of the square, and there are symmetry relations $\theta_{j,-1} = \theta_{j,1}$ and $\theta_{j,k} = \theta_{k,j}$ for simplifying the equations for points on the lines of symmetry. For example,

$$\left.\begin{aligned}
&\text{for } j = k = 0, && \frac{\partial\theta_{0,0}}{\partial t} = D(\theta_{0,0})\,\frac{4(\theta_{0,1}-\theta_{0,0})}{(\delta s)^2},\\
&\text{for } j \neq 0,\ k = 0, && \frac{\partial\theta_{j,0}}{\partial t} = D(\theta_{j,0})\,\frac{(\theta_{j+1,0}+\theta_{j-1,0}+2\theta_{j,1}-4\theta_{j,0})}{(\delta s)^2},\\
&\text{for } j = k \neq 0, && \frac{\partial\theta_{j,j}}{\partial t} = D(\theta_{j,j})\,\frac{2(\theta_{j+1,j}+\theta_{j-1,j}-2\theta_{j,j})}{(\delta s)^2}.
\end{aligned}\right\} \tag{3·13}$$

For $n = 2$, equations (3·13) cover all possibilities.

(*d*) *Two-dimensional non-radial heat flow in a circular cylinder.* A similar treatment can be applied to heat flow in a circular cylinder without the restriction to axial symmetry.

The equation of heat conduction in plane polar co-ordinates (r, ϕ) is now

$$\frac{\partial\theta}{\partial t} = D(\theta)\left(\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\theta}{\partial\phi^2}\right). \tag{3·14}$$

For the r-derivatives the same approximation can be used as in the treatment of the axially symmetrical case (*b*) above. Further, if the range 2π of the azimuth ϕ is also divided into equal intervals $\delta\phi$, the approximation (3·2) can be applied to the ϕ-derivative in (3·14). If $\theta_{j,k}$ now stands for the temperature at the point $r = j\,\delta r$, and $\phi = k\,\delta\phi$, the finite difference form of (3·14) for $j \neq 0$ is then

$$\frac{\partial\theta_{j,k}}{\partial t} = \frac{D(\theta_{j,k})}{(\delta r)^2}\left[\frac{(2j+1)\,\theta_{j+1,k}-4\theta_{j,k}+(2j-1)\,\theta_{j-1,k}}{2j}+\frac{1}{j^2}\,\frac{\theta_{j,k+1}-2\theta_{j,k}+\theta_{j,k-1}}{(\delta\phi)^2}\right]. \tag{3·15}$$

This is not applicable to the temperature at the centre $j = 0$. But for $\delta\phi = \frac{1}{2}\pi$, the configuration of the points of the polar co-ordinate lattice on $r = \delta r$ (that is, $j = 1$) and at the centre is exactly the same as for a Cartesian lattice, so equation (3·12) can be taken over directly as the appropriate equation in the cylindrical case with $\delta\phi = \frac{1}{2}\pi$. With the suffixes appropriate to the polar co-ordinate lattice, it becomes

$$\frac{\partial\theta_0}{\partial t} = D(\theta_0)\left[\frac{\theta_{1,0}+\theta_{1,1}+\theta_{1,2}+\theta_{1,3}-4\theta_0}{(\delta r)^2}\right]. \tag{3·16}$$

More generally, let $\bar{\theta}_1$ be the mean temperature on the circle $r = \delta r\,(j = 1)$. The difference between this mean temperature and the temperature θ_0 at the centre is

$$\bar{\theta}_1-\theta_0 = \tfrac{1}{4}(\delta r)^2\,(\nabla^2\theta)_0+O(\delta r)^4, \tag{3·17}$$

and whatever the value of $\delta\phi$, the best available value of $\bar{\theta}_1$ is the arithmetic mean of the values of θ on the circle $r = \delta r$. Use of (3·17) for $(\nabla^2\theta)_0$ in the heat conduction equation (1·1) then gives

$$\frac{\partial\theta_0}{\partial t} = \frac{4D(\theta_0)}{(\delta r)^2}(\bar{\theta}_1 - \theta_0); \tag{3·18}$$

for $\delta\phi = \frac{1}{2}\pi$ this gives (3·16). For axial symmetry, the temperature is uniform over the circle $r = \delta r$, and then $\bar{\theta}_1 = \theta_1$ and (3·18) reduces to (3·8), providing an alternative deduction of the latter formula.

The practical value of this method of evaluating solutions of the equation of heat conduction depends on the number of intervals into which the space range has to be divided in order to attain results of acceptable accuracy, bearing in mind the limited accuracy of the data on the thermal properties (see end of § 2); the smaller this number is, the more quickly and easily can solutions be evaluated.

This number depends on the accuracy required and on the numerical magnitude of the errors resulting from the use of approximations (3·2) and (3·3). This can be tested in two ways: firstly, by taking cases in which both the partial differential equation and the set of ordinary differential equations by which it is replaced have analytical solutions, from which numerical results for particular examples can be evaluated and compared; and secondly, by evaluating two solutions, with the same surface conditions, with the space range divided into two different numbers of intervals; from comparison of such solutions the errors in each can be estimated. Both methods have been used in the present work; results are discussed in §§ 7 and 14 respectively.

It should be noted that these errors, due to the use of the analytical approximations (3·2) and (3·3), can *not* be determined satisfactorily by comparison of calculated and experimentally observed transient temperature distribution. As will be seen later (§ 14), the errors due to the analytical approximations alone are often smaller than the possible discrepancies due to uncertainty in the exact conditions prevailing in the experiments (e.g. departures from radial heating, in the case of a cylinder, or departures from nominal values of thermal properties), and to experimental errors in the difficult technique required for the measurement of time-varying temperatures in the interior of masses of material at high temperatures.

4. TREATMENT OF BOUNDARY CONDITIONS OTHER THAN GIVEN TIME VARIATIONS OF SURFACE TEMPERATURE

When the boundary conditions at the surfaces of the solid through which heat conduction is being considered have the form of given time variations of surface temperatures, then these data go directly into the appropriate equations of the set (3·4) (for the slab and similarly for other geometrical shapes). But an extension of the equations is required to deal with boundary conditions involving the surface heat transfer explicitly. A similar extension is required in treating the heat flow across the surface of separation of two dissimilar materials.

The surface heat flux (i.e. surface heat transfer per unit area and per unit time) will be written F and will be taken positive in the direction of x or r increasing. Its distribution over the surface may be given as a function of time only, but will more often be related to the surface temperature in a definite way, which may or may not depend on the time. For example, for Newton's law of cooling, it is proportional to $(\theta_s - \theta_a)$, where θ_a is the ambient temperature, and for radiative heat transfer between a source at temperature θ_f (which may depend on the time) and the surface, the surface heat flux is proportional to $(\theta_f^4 - \theta_s^4)$. When necessary to emphasize the variables in terms of which the surface heat flux is given, they will be written explicitly as arguments of F, in general as $F(\theta_s, t)$. The temperature distribution in the solid is related to the heat flux by

$$K(\theta_s)\left(\frac{\partial\theta}{\partial x}\right)_s = -F(\theta_s, t) \tag{4·1}$$

for the slab, and

$$K(\theta_s)\left(\frac{\partial\theta}{\partial r}\right)_s = -F(\theta_s, t) \tag{4·2}$$

for the cylinder. The only new question is how to deal with the surface temperature gradient in terms of the approximation used in § 3. The argument will be presented for the plane case; its extension to the cylindrical case is obvious, and the results for this case will be quoted, without details of their derivation.

One method of dealing with the surface temperature gradient is, in effect, to extrapolate its value from the temperatures $\theta_n = \theta_s$, θ_{n-1} and θ_{n-2} at the surface and at one and two intervals inwards from it. If $\partial^3\theta/\partial x^3$ is assumed to be negligible, then

$$\left.\begin{aligned}\theta_{n-1} &= \theta_s - \left(\frac{\partial\theta}{\partial x}\right)_s(\delta x) + \frac{1}{2}\left(\frac{\partial^2\theta}{\partial x^2}\right)_s(\delta x)^2,\\ \theta_{n-2} &= \theta_s - 2\left(\frac{\partial\theta}{\partial x}\right)_s(\delta x) + 2\left(\frac{\partial^2\theta}{\partial x^2}\right)_s(\delta x)^2,\end{aligned}\right\} \tag{4·3}$$

so, to this approximation,

$$\left(\frac{\partial\theta}{\partial x}\right)_s = \frac{\theta_{n-2} - 4\theta_{n-1} + 3\theta_s}{2(\delta x)}. \tag{4·4}$$

It would be possible to include further terms in the Taylor series (4·3), but this would in effect only increase the range from which the extrapolation of the surface temperature gradient is carried out, and it is doubtful if the result would be much better. On the other hand, it certainly seems worth while using a better approximation than the crude expression $(\theta_s - \theta_{n-1})/(\delta x)$, by taking the Taylor series as far as the terms involving $(\partial^2\theta/\partial x^2)_s$ and eliminating this term, since all non-steady heat flow depends on this derivative being appreciably different from zero.

Combination of (4·1) and (4·4) gives an algebraic equation for the surface temperature in terms of the temperatures θ_{n-1} and θ_{n-2}, which are given by integration of (3·4) for $j = n-1$ and $j = n-2$ respectively. This equation is

$$2\delta x[F(\theta_s, t)/K(\theta_s)] + 3\theta_s = 4\theta_{n-1} - \theta_{n-2}, \tag{4·5}$$

and applies to both plane and cylinder.

The other method of dealing with the surface temperature gradient is to imagine the heat transfer by conduction carried one interval beyond the real surface $x = x_s$ to $x_{n+1} = x_s + \delta x$, so that

$$\frac{\partial\theta_s}{\partial t} = D(\theta_s)\,\frac{\theta_{n+1} - 2\theta_s + \theta_{n-1}}{(\delta x)^2} \tag{4·6}$$

(note, $\theta_n = \theta_s$). From (3·3) the temperature gradient at the position of the real surface would be, to the same order of approximation,

$$\left(\frac{\partial\theta}{\partial x}\right)_s = \frac{\theta_{n+1}-\theta_{n-1}}{2\,\delta x}. \tag{4·7}$$

Then imagine the temperature θ_{n+1} to be controlled so that the relation between $(\partial\theta/\partial x)_s$, given by (4·7), and θ_s is that specified by (4·1). Such control might not correspond to anything that could be realized in practice, but the heat flow inside the real surface, for adequately specified conditions at that surface, cannot depend on the means outside that surface by which these conditions are attained. Elimination of θ_{n+1} between (4·6) and (4·7) gives a *differential* equation for the surface temperature, namely

$$\frac{\partial\theta_s}{\partial t} = -\frac{2}{\delta x}D(\theta_s)\left[\frac{F(\theta_s, t)}{K(\theta_s)}+\frac{\theta_s-\theta_{n-1}}{\delta x}\right], \tag{4·8}$$

instead of the *algebraic* equation (4·5).

The approximation (4·7) is better than (4·4) and has the advantage of being in effect an interpolated rather than an extrapolated value of the surface temperature gradient, and use of (4·8) should give more accurate results than use of (4·5). The two terms in (4·8) can be interpreted, roughly, as the contribution to the heating of the surface by the flow of heat into it from outside and the (negative) contribution due to the flow of heat away from it by conduction to the inside.

In particular, if there is no surface heat transfer, the surface temperature is given by

$$\frac{\partial\theta_s}{\partial t} = \frac{2}{(\delta x)^2}D(\theta_s)\,(\theta_{n-1}-\theta_s). \tag{4·9}$$

Equation (4·8) applies if the material to which D refers lies to the left ($x<x_s$) of the surface. Similarly, for a material lying to the right ($x>x_s$) of the surface,

$$\frac{\partial\theta_s}{\partial t} = -\frac{2}{\delta x}D(\theta_s)\left[-\frac{F(\theta_s, t)}{K(\theta_s)}+\frac{\theta_s-\theta_{n+1}}{\delta x}\right]. \tag{4·10}$$

A particular application of (4·8) and (4·10) is to the conditions at the surface of separation $x = x_s$ of two different conducting materials in contact. Let suffixes a and b refer to the materials to the left and right of x_s; δx may be taken as different in the two materials. The boundary conditions are that the surface temperature and surface heat flux are continuous across the boundary, that is

$$\theta_{sa} = \theta_{sb}, \quad F_a(\theta_s, t) = F_b(\theta_s, t). \tag{4·11}$$

For the material to the left of the surface, (4·8) applies with $D = D_a$, $\delta x = (\delta x)_a$; for that to the right, (4·10) applies with $D = D_b$, $\delta x = (\delta x)_b$. Elimination of the surface temperature gradient from these two equations by use of the boundary conditions (4·11) gives

$$\frac{1}{2}\left[\frac{K_b(\delta x)_b}{D_b}+\frac{K_a(\delta x)_a}{D_a}\right]\frac{\partial\theta_s}{\partial t} = \left[\left\{\frac{K_b}{(\delta x)_b}\right\}(\theta_{n+1}-\theta_s)-\left\{\frac{K_a}{(\delta x)_a}\right\}(\theta_s-\theta_{n-1})\right]. \tag{4·12}$$

For the cylinder, the corresponding equation to (4·8) is

$$\frac{\partial\theta_s}{\partial t} = -\frac{2}{\delta r}D(\theta_s)\left[\left(1+\frac{1}{2n}\right)\frac{F(\theta_s, t)}{K(\theta_s)}+\frac{\theta_s-\theta_{n-1}}{\delta r}\right], \tag{4·13}$$

for the outer surface of a solid or hollow cylinder (radius $n\delta r$), and

$$\frac{\partial \theta_s}{\partial t} = \frac{2}{\delta r} D(\theta_s) \left[\left(1 - \frac{1}{2m}\right) \frac{F(\theta_s, t)}{K(\theta_s)} - \frac{\theta_s - \theta_{m+1}}{\delta r} \right] \tag{4·14}$$

for the inner surface of a hollow cylinder (internal radius $m\delta r$). If there is not heat transfer at the outer or inner surface, the equation corresponding to (4·9) is

$$\frac{\partial \theta_s}{\partial t} = \frac{2}{(\delta r)^2} D(\theta_s)\,(\theta_{n-1} - \theta_s) \quad \text{or} \quad \frac{2}{(\delta r)^2} D(\theta_s)\,(\theta_{m+1} - \theta_s). \tag{4·15}$$

respectively.

A better approximation still than (4·7) is

$$\left(\frac{\partial \theta}{\partial x}\right)_s = \frac{2\theta_{n+1} + 3\theta_s - 6\theta_{n-1} + \theta_{n-2}}{6\delta x},$$

and combination of this with (4·1) and (4·6) gives

$$\frac{\partial \theta_s}{\partial t} = \frac{3}{\delta x} D(\theta_s) \left[\frac{F(\theta_s, t)}{K(\theta_s)} + \frac{7\theta_s - 8\theta_{n-1} + \theta_{n-2}}{6\delta x} \right]. \tag{4·16}$$

Corresponding equations can be obtained for the cylindrical case.

PART II. THERMAL PROPERTIES INDEPENDENT OF TEMPERATURE

5. Reduction to dimensionless measures of length and time

As already explained, some preliminary work for the case of constant diffusivity was done for three main purposes: firstly, to gain experience in the use of the differential analyser in work of this kind; secondly, to examine the numerical magnitude of the errors introduced by the replacement of the space derivative by the corresponding finite difference and hence to determine the smallest number of intervals into which the space range could be divided so as to give results of acceptable accuracy; and thirdly, to test the accuracy of the differential analyser when applied to the resulting equations.

When the diffusivity is constant, it is convenient to work in terms of non-dimensional measures of length and time. If $2l$ is the thickness of the slab, a the radius of the cylinder, $2a$ the side of the square,

$$X = x/l, \quad R = r/a, \quad \text{and} \quad X = x/a, \quad Y = y/a, \tag{5·1}$$

in the three cases respectively, are convenient dimensionless measures of length, and

$$\tau = Dt/l^2 \quad \text{or} \quad Dt/a^2, \tag{5·2}$$

respectively, are convenient dimensionless measures of time. Use of these dimensionless measures of length and time has a further advantage that the results apply equally to slabs of any thickness and cylinders of any radius. In terms of X, τ the partial differential equation (1·1) reduces to

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial X^2}, \tag{5·3}$$

and the approximate equation (3·4) becomes

$$\frac{\partial \theta_j}{\partial \tau} = n^2 (\theta_{j+1} - 2\theta_j + \theta_{j-1}), \tag{5·4}$$

where $n = l/\delta x$ is the number of intervals δx in the half thickness l of the slab. If

$$T = n^2 \tau = n^2 Dt/l^2 = Dt/(\delta x)^2, \tag{5·5}$$

(5·4) can be written

$$\frac{\partial \theta_j}{\partial T} = \theta_{j+1} - 2\theta_j + \theta_{j-1}, \tag{5·6}$$

which is convenient for analytical work and for work on the differential analyser. But since the measure T of time depends on n, which is a property of the approximation used for the solution of the heat-flow equations and not of the physical process of heat conduction, it has not the same physical significance as τ.

Similarly, for radial heat flow in a cylinder let

$$T = n^2 \tau = n^2 Dt/a^2 = Dt/(\delta r)^2; \tag{5·7}$$

then (3·7) and (3·8) become

$$\frac{\partial \theta_j}{\partial T} = \frac{(2j+1)(\theta_{j+1} - \theta_j) - (2j-1)(\theta_j - \theta_{j-1})}{2j} \quad (j > 0), \tag{5·8}$$

$$\frac{\partial \theta_0}{\partial T} = 4(\theta_1 - \theta_0). \tag{5·9}$$

6. Application of the differential analyser

The application of the differential analyser to equations (5·6) for the plane case, or (5·8) and (5·9) for the cylindrical case, is straightforward.

For symmetrical heat flow in a slab, or for a cylinder, n integrators are required, and one input table for supplying to the machine the time variation of the surface temperature.

For the plane case, it is most convenient to take (5·6) in the form

$$\frac{\partial \theta_j}{\partial T} = (\theta_{j+1} - \theta_j) - (\theta_j - \theta_{j-1}). \tag{6·1}$$

Each difference of two adjacent temperatures, such as $\theta_{j+1} - \theta_j$, occurs in two equations; for example, $\theta_{j+1} - \theta_j$ occurs in the equations for θ_j and for θ_{j+1}. It is convenient to form these differences and to combine them so as to form the right-hand sides of equations (6·1), as two steps. The output from the integrator which evaluates θ_j by integration of $\partial \theta_j / \partial T$ is combined, by two adding units, with the outputs from the integrators on either side, giving $\theta_{j+1} - \theta_j$ and $\theta_j - \theta_{j-1}$. Then $\theta_{j+1} - \theta_j$ is combined with $\theta_{j+2} - \theta_{j+1}$ to give $\partial \theta_{j+1} / \partial T$, and with $\theta_j - \theta_{j-1}$ to give $\partial \theta_j / \partial T$.

For the point one layer in from the surface, that is, for $x = x_{n-1}$,

$$\frac{\partial \theta_{n-1}}{\partial T} = (\theta_n - \theta_{n-1}) - (\theta_{n-1} - \theta_{n-2}),$$

and θ_n is the surface temperature θ_s, which is fed in from an input table on which a curve of θ_s against T, previously drawn, is placed.

For the cylindrical case, the only difference is that, as indicated by the equations in the form (5·8), the differences $(\theta_{j+1}-\theta_j)$ and $(\theta_j-\theta_{j-1})$ between the outputs from successive integrators have to be multiplied by numerical factors before being combined to form $\partial\theta_j/\partial T$. These factors are independent of T and in practical cases, when n is fairly small, they are ratios of small integers and can easily be introduced by gears.

The time required for a machine solution depends on the range of T to be covered. Formula (5·5) shows that, for a given period of time t, the range of T is proportional to n^2, so that the running time for a solution is proportional to n^2 as already mentioned.

7. Results and discussion

Examination of the errors introduced by the use of the analytical approximations (3·2) and (3·3) to reduce the partial differential equation of heat flow to a set of ordinary differential equations, and of the accuracy of the differential analyser for the solution of this set of equations, can best be done for the case of constant diffusivity. In this case both the partial differential equation and the set of ordinary differential equations have analytical solutions which, for suitably chosen time variations of surface temperature, are not too inconvenient for numerical evaluation. If then these analytical solutions are evaluated, and a solution of the set of ordinary differential equations is also obtained on the differential analyser, all for the same time variation of surface temperature, comparison of the three sets of results will enable a clear distinction to be made between the errors introduced by the use of the analytical approximation (3·2) and errors in the solution of the resulting equations by the differential analyser.

The analytical solutions, both of the partial differential equation and the set of ordinary differential equations, is simpler for the plane case than for the cylinder or square prism, so it is much easier to make the tests for the plane case. If use of the equations (3·4) and (3·5) for the plane case is satisfactory, it seems safe to assume that the corresponding use of (3·7) and (3·8) for the cylindrical case, and (3·12) and (3·13) for the square prism, will also be satisfactory.

For this reason these tests are restricted to the plane case. For such a test it is necessary to choose a variation of surface temperatures θ_s with time, so that analytical solutions, both of the exact equation (5·3) and of the approximate equations (5·6), could be obtained, and, moreover, in forms suitable for numerical evaluation. Three cases were taken, namely,

Case I. $\theta_s = 0$ for $T<0$, $\theta_s = 1$ for $T>0$.

Case II. $\theta_s = T$ for $T>0$.

Case III. $\theta_s = \frac{1}{3}[50-e^{-T/5}(T^2+10T+50)]$ for $T>0$.

Case I is an extreme case in which the surface temperature is suddenly increased; this is more drastic than any case likely to occur in practice, since even when a cold mass of steel is suddenly placed in a furnace, the surface temperature does not change discontinuously. Case II is that in which the surface temperature rises at constant rate. Case III was chosen to be representative of the general shape of heating curves of ingots in furnaces. In all cases the temperature was taken as initially uniform throughout the slab, and temperatures were reckoned from this initial temperature as zero.

It is obvious from the equations that as long as the diffusivity is constant, the scale of temperature is arbitrary, so case I covers all cases in which the surface temperature is suddenly raised. Similarly, in case II a single solution covers all values of rate of rise of temperature.

The analytical treatment of the heat flow in these cases, both for the exact solution (5·3) and for the approximate equations (5·6), is given in the appendix.

The results for the solution of the exact equations (5·3), for the analytical solution of the approximate equations (5·6) with $n = 6$, and for the differential analyser solution of these same equations, are given in table 1.

These results are most satisfactory, both from the point of view of the accuracy attainable by the use of the approximate equations with a comparatively small number n of intervals, and from the point of view of the errors of the differential analyser solution of the approximate equations.

Case I is extremely drastic from the point of view of the essential approximation (3·2) on which equations (5·6) are based. This approximation is certainly not valid for very small times when the surface temperature is 1 and the internal temperature even at a single interval δx from the surface is still inappreciably different from zero. Without reference to the detailed results of table 1, it can easily be seen from graphical considerations that under such circumstances the use of this approximation will make the internal temperature begin to rise too fast; and it can be seen from table 1 that this is what does occur. But the error in the central temperature is never more than about $\frac{1}{2}$ % of the surface rise of temperature, that is, about 5° for 1000° rise of surface temperature; and by the time the central temperature has risen to half the surface temperature, the error is down to about 1° per 1000° rise of surface temperature. The accuracy of the differential analyser in the solution of the approximate equations is better than the accuracy of these equations themselves.

The values of θ_0 for the analytical solution (both for the exact and approximate equations) in case II are the time integrals of the values of θ_0 in case I, and the same therefore applies to the errors in the results calculated from the solution of the approximate equations. At short times, the error in the central temperature is a substantial fraction of that temperature, which is itself small, and it seems more significant to compare this error with the instantaneous value of the *surface* temperature; the maximum errors in the ratio of θ_0 to the *instantaneous* surface temperature is 0·4 %.

Now in view of the variation of the thermal properties, at a given temperature, between different specimens of nominally the same steel (see end of § 2), it seems unlikely that the thermal properties of any particular specimen of steel will be known to a greater accuracy than 2 % at best. From the results of table 1 it is clear that, in view of this, the accuracy given by the use of $n = 6$ in the approximation is unnecessarily high, and that the accuracy of the differential analyser in the solution of these equations is ample.

The encouraging results obtained with $n = 6$ suggested that adequate accuracy for many practical purposes could be obtained by taking a smaller number of intervals, $n = 4$ or $n = 3$, or even $n = 2$. If this were possible, it would simplify the machine set-up for the slab and cylinder and would considerably shorten the running time as explained at the end of § 6; it would also bring the square prism within the practical range of solution by the machine.

A test was therefore made for the plane case with constant rate of rise of surface temperature using only three intervals from centre to surface of the slab. The three-interval solution was evaluated from the analytical results of the Appendix, to avoid confusing errors due to the analytical approximations with any which might be introduced in the course of integration on the differential analyser.

TABLE 1

Case I. $\theta_s = 1$

		central temperature θ_0		
			finite difference equation ($n=6$)	
$T = 36\tau$ $= 36Dt/l^2$	surface temp. θ_s	partial differential equation	analytical solution	differential analyser solution
2	1·000	0·005	0·009	0·009
4	1·000	0·068	0·074	0·074
6	1·000	0·167	0·171	0·171
8	1·000	0·267	0·270	0·270
10	1·000	0·359	0·361	0·361
12	1·000	0·441	0·442	0·443
14	1·000	0·512	0·513	0·514
16	1·000	0·575	0·575	0·576
18	1·000	0·629	0·629	0·631
20	1·000	0·677	0·676	0·679
22	1·000	0·718	0·717	0·720
24	1·000	0·754	0·753	0·757

Case II. $\theta_s = T$

		central temperature θ_0		
			finite difference equation ($n=6$)	
$T = 36\tau$	surface temp.	partial differential equation	analytical solution	differential analyser solution
2	2	0·001	0·002	0·004
4	4	0·065	0·077	0·076
6	6	0·292	0·320	0·318
8	8	0·730	0·763	0·761
12	12	2·16	2·20	2·19
16	16	4·20	4·24	4·22
20	20	6·72	6·76	6·73
24	24	9·59	9·62	9·59
28	28	12·73	12·76	12·72
32	32	16·07	16·10	16·05
40	40	23·20	23·22	23·15

Case III. $\theta_s = \frac{1}{3}[50 - (T^2 + 10T + 50)\, e^{-T/5}]$

		central temperature θ_0		
			finite difference equation ($n=6$)	
$T = 36\tau$	surface temp.	partial differential equation	analytical solution	differential analyser solution
4	0·79	0·00	0·01	0·00
16	10·33	2·01	2·03	2·03
32	15·90	9·39	9·39	9·35
50	16·62	14·30	14·29	14·25

Table 2 shows the results, giving at three different points through the slab a comparison between the exact solution, the analytical three-interval solution, the analytical six-interval solution and the differential analyser six-interval solution.

Table 2. Case II. $\theta_s = 36\tau_s (= T \text{ for } n = 6)$

		finite difference equations		
		analytical solution		
36τ $= 36Dt/l^2$	partial differential equation	$n = 3$	$n = 6$	differential analyser solution $n = 6$
temperature at $x = \frac{1}{3}l$				
2	0·02	0·05	0·03	0·03
4	0·23	0·30	0·25	0·25
6	0·66	0·76	0·69	0·69
8	1·30	1·41	1·33	1·32
12	3·07	3·20	3·10	3·09
16	5·37	5·51	5·41	5·39
20	8·09	8·21	8·12	8·10
24	11·10	11·22	11·13	11·11
28	14·36	14·46	14·39	14·35
32	17·80	17·89	17·82	17·78
40	25·04	25·10	25·05	25·00
temperature at $x = \frac{2}{3}l$				
2	0·30	0·37	0·32	0·32
4	1·12	1·19	1·14	1·14
6	2·17	2·24	2·19	2·19
8	3·37	3·45	3·39	3·39
12	6·08	6·16	6·10	6·09
16	9·10	9·18	9·12	9·11
20	12·36	12·43	12·38	12·36
24	15·79	15·86	15·81	15·79
28	19·36	19·42	19·38	19·36
32	23·04	23·09	23·05	23·02
40	30·60	30·64	30·61	30·57
central temperature				
2	0·00	0·01	0·002	0·004
4	0·065	0·12	0·077	0·076
6	0·292	0·39	0·320	0·318
8	0·730	0·85	0·763	0·761
12	2·16	2·31	2·20	2·19
16	4·20	4·35	4·24	4·22
20	6·72	6·86	6·76	6·73
24	9·59	9·72	9·62	9·59
28	12·73	12·84	12·76	12·72
32	16·07	16·17	16·10	16·05
40	23·20	23·27	23·22	23·15

The results are most encouraging. The maximum error in the central temperature in the three-interval solution is about $1\frac{1}{2}\%$ of the instantaneous surface temperature (for the six-interval solution the maximum error is about $0·4\%$, so that the error is about proportional to $1/n^2$ as would be expected), and the maximum error elsewhere is not much greater.

These results suggest that, at any rate for rough results, a two-interval or even a one-interval solution might be adequate. For $n = 1$, equation (5·6) for the plane case becomes just

$$\frac{\partial \theta_0}{\partial T} = 2(\theta_s - \theta_0), \tag{7·1}$$

and equation (5·9) for the cylinder becomes just

$$\frac{\partial \theta_0}{\partial T} = 4(\theta_s - \theta_0), \tag{7·2}$$

and the latter also applies to a square prism.

The conclusions from this work are firstly, that the analytical approximation involved in the replacement of the space derivative by a corresponding finite difference is adequate for practical purposes even when the half-thickness of the slab or radius of the cylinder is divided into not more than three, or perhaps even two, intervals and secondly, that use of the differential analyser is practicable, and its accuracy ample, for handling the equations arising from this analytical approximation.

PART III. THERMAL PROPERTIES VARYING WITH TEMPERATURE

8. THE EQUATIONS OF HEAT CONDUCTION WHEN THERMAL PROPERTIES VARY WITH TEMPERATURE

As already mentioned, the thermal properties of steel vary considerably with temperature, and one of the main objects of the present work was the development of means of evaluating solutions of the equations of heat conduction in which this variation of thermal properties could be adequately taken into account.

The thermal properties of steel are not even functions of the temperature only, but depend on the way it is and has been changing. This is shown, for example, by the fact that the transformation which occurs in the neighbourhood of 720° C is observed to take place at a temperature which depends on the previous thermal history of the material, and not at a fixed temperature.

The present work, however, is mainly concerned with the treatment of the variation of thermal properties in so far as they can be regarded as functions of the temperature only, and this restriction will be understood unless the converse is explicitly stated. In practice, this restriction is not a severe one in the present state of knowledge, since the data at present available on the thermal properties of steel are inadequate to enable quantitative calculations to be made on any more accurate basis, and a practical method of taking into account continuously the variation of the thermal properties with temperature is already a very substantial advance on previously available possibilities. An outline of some possibilities of a more accurate treatment if adequate quantitative data were available is given at the end of § 10.

The equations of heat conduction themselves need some consideration first.

When the thermal conductivity varies with the temperature, the equation of heat conduction is no longer (1·1), but

$$\rho\sigma \frac{\partial \theta}{\partial t} = \operatorname{div}(K \operatorname{grad} \theta). \tag{8·1}$$

For a homogeneous substance for which the conductivity varies with temperature, this can be written

$$\rho\sigma \frac{\partial \theta}{\partial t} = K\nabla^2\theta + \frac{dK}{d\theta}(\operatorname{grad}\theta)^2. \tag{8·2}$$

However, in practical cases the second term in (8·2) is much smaller than the first, except perhaps in the earlier stages of heat flow in a quench; this has been verified by rough evaluation of the terms in some typical cases. Even in quenching, the second term in (8·2) may still be small compared with the first, since $\nabla^2\theta$ will be large where and when $(\text{grad}\,\theta)$ is large; but this has not been examined quantitatively.* On these grounds it seems that omission of the second term in (8·2) will give an adequate approximation. Then (8·2) reduces to (1·1) with $D = K/\rho\sigma$ a function of temperature.

Although for most practical purposes it seems that the neglect of the second term in (8·2) gives an adequate approximation, an alternative form which avoids this approximation should be noted.

Let K_0 be the conductivity at some standard temperature, and let Θ be a function of temperature defined by†

$$\Theta = \int_0^\theta (K/K_0)\, d\theta; \tag{8·3}$$

Θ can be regarded as a modified temperature, measured on a scale defined by (8·3). Then

$$\frac{\partial\Theta}{\partial t} = \frac{K}{K_0}\frac{\partial\theta}{\partial t}, \quad \text{grad}\,\Theta = \frac{K}{K_0}\text{grad}\,\theta,$$

so (8·1) becomes

$$\frac{\partial\Theta}{\partial t} = D\nabla^2\Theta, \tag{8·4}$$

where $D = K/\rho\sigma$; this can be regarded as a known function of Θ. Thus it is strictly the modified temperature Θ which satisfies the simple form (1·1) of the heat-conduction equation. This form (8·4) of the equation has not actually been used in the present work, as it was only recognized at a late stage, and the approximation of neglecting the $dK/d\theta$ term in (8·2) seemed adequate; but it might be advisable to use (8·4) in preference to (1·1) in the earlier stages, at least, of the heat flow in a quench, when the instantaneous range of temperature, and so of the conductivity, through the material is considerable.

Another form of this equation is sometimes useful. If $H(\theta)$ is the heat capacity per unit mass at temperature θ, namely,

$$H(\theta) = \int_\alpha^\theta \sigma\, d\theta, \tag{8·5}$$

where α is some convenient standard temperature (taken as 50° C in table 45 of the *Second Report of the Alloy Steels Research Committee* (1939), p. 234, then

$$(1/D)\,\partial\Theta/\partial t = (\rho/K_0)\,\partial H/\partial t,$$

so (8·4) can be written

$$(\rho/K_0)\,\partial H/\partial t = \nabla^2\Theta. \tag{8·6}$$

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* *Note added in proof, April* 1946. Since this paper was written, a considerable amount of further work on the lines given in this paper has been done, including an extensive application to heat flow in quenching. This work has shown that in quenching it is quite possible for the $dK/d\theta$ term in (8·2) to be of the same order of magnitude as the other terms, and that its neglect may, in practical cases, result in errors of the order of 100° C in the internal temperature at a definite time after the beginning of the quench. An account of this further work will be published separately.

† K_0 is only introduced in order that Θ may have the physical dimensions of temperature and a magnitude of the same order as that of the temperature itself.

From a table of K as a function of θ, the relation between θ and Θ can be evaluated, and from this and a table of $H(\theta)$, the relation between H and Θ can be constructed.

A modification of this form of the heat-conduction equation, involving the neglect of the second term in (8·2), has been used in parts of the present work. If $Q(\theta)$ is a function of temperature defined by

$$Q(\theta)=\rho\int_{\alpha}^{\theta}(\sigma/K)\,d\theta=\rho\int_{H(\alpha)}^{H(\theta)}(1/K)\,dH, \tag{8·7}$$

equation (1·1) can be written

$$\frac{\partial Q}{\partial t}=\nabla^2\theta, \tag{8·8}$$

or for heat flow in one dimension

$$\frac{\partial Q}{\partial t}=\frac{\partial^2\theta}{\partial x^2}. \tag{8·9}$$

From tables of σ (or H) and K as functions of θ, the function $Q(\theta)$ can be calculated.

In this work, the variation of the density ρ with temperature, arising from thermal expansion, has been neglected. The treatment could be extended to include this variation if required, but it should be noted that if thermal expansion is taken into account through its effect on the density factor in the diffusivity, it must also be taken into account in its effect on the thickness of the slab or cylinder, that is, on the space range over which the integration is to be carried. This cancels two-thirds of its effect on the density, so that inclusion of its effect on the density *only* is a worse approximation than neglecting it altogether. Its effect is greater at high temperature and, in view of the uncertainty in the thermal properties at high temperatures, its neglect seems an adequate approximation.

Although the thermal properties of steel can be represented much more closely by regarding them as functions of temperature only than by taking them as constant, the abnormal thermal properties of steel at temperatures in the neighbourhood of the transformation temperature need further consideration if details of the heat flow and temperature distribution in that range are required. The phenomena associated with this transformation will be discussed in § 10.

9. APPROXIMATE TREATMENT OF THE EQUATION WITH VARIABLE THERMAL PROPERTIES

The approximations of § 3 can be applied directly to the equation of heat conduction in the form (1·1), (8·4), (8·6) or (8·8). For example, for one-dimensional heat flow, use of (3·2) in the heat-conduction equation in the form (1·1) gives the set of equations (3·4). Use of the same approximation in (8·4) gives

$$\frac{\partial\Theta_j}{\partial t}=D(\Theta_j)\,\frac{\Theta_{j+1}-2\Theta_j+\Theta_{j-1}}{(\delta x)^2}, \tag{9·1}$$

its use in (8·6) and in (8·8) gives

$$\frac{\rho}{K_0}\frac{\partial H_j}{\partial t}=\frac{\Theta_{j+1}-2\Theta_j+\Theta_{j-1}}{(\delta x)^2}. \tag{9·2}$$

and

$$\frac{\partial Q_j}{\partial t}=\frac{\theta_{j+1}-2\theta_j+\theta_{j-1}}{(\delta x)^2}, \tag{9·3}$$

respectively, and similarly for the equations appropriate to the cylinder and square prism. Application of any of these forms of the equation of heat conduction enables the thermal properties appropriate to the temperature at each point of the solid to be used in evaluating the solutions of that equation. Equations (9·1) and (9·2) only involve the approximation (3·2); equations (3·4) and (9·3) involve in addition the neglect of the $dK/d\theta$ term in (8·1).

A further approximation, which considerably simplifies the details of evaluating solutions, and which in many cases gives results of adequate accuracy for practical purposes, is the use of a diffusivity uniform through the material at each instant, but varying with the time.

As already mentioned, in most heat treatments other than quenching, the range of temperature through the steel at any moment is never very large; further, except in the neighbourhood of the transformation temperature the thermal properties do not vary very rapidly with temperature, so that it may often be an adequate approximation to take the diffusivity as uniform through the steel at each instant, with a value appropriate to some instantaneous mean temperature $\bar{\theta}$, though in the course of time this mean temperature and the values of the thermal properties at that temperature, may vary over a considerable range. With this further approximation, equation (1·1) becomes

$$\frac{\partial\theta}{\partial t} = D(\bar{\theta})\nabla^2\theta, \tag{9·4}$$

or for one-dimensional heat flow

$$\frac{\partial\theta}{\partial t} = D(\bar{\theta})\frac{\partial^2\theta}{\partial x^2}. \tag{9·5}$$

If the way in which the mean temperature is to be taken is specified, then $D(\bar{\theta})$ is a definite function of time, though not a known function until the solution has been carried out (unless the surface temperature θ_s is taken as an adequate approximation to the appropriate mean temperature $\bar{\theta}$). Then for the one-dimensional case it is convenient to define dimensionless measures τ, X of time and distance by

$$\tau = (1/l^2)\int D(\bar{\theta})\,dt \tag{9·6}$$

and $X = x/l$ so that (9·5) reduces to the standard form, dimensionless in x and t,

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial X^2}$$

[compare (5·3)]. The quantity τ can be thought of as a measure of time on a scale distorted according to the value of $D(\bar{\theta})$ at each moment and (to the approximations involved in reducing the equation to the form (9·5)) heat flow on the distorted τ scale of time is exactly similar to heat flow in a material of constant diffusivity on the ordinary time scale. Application of the approximation (3·2) for the space derivative then gives the set of equations (5·6), with T now given by

$$T = n^2\tau = (n^2/l^2)\int D(\bar{\theta})\,dt. \tag{9·7}$$

Thus use of the approximation of a uniform diffusivity at each instant, and of the T-scale of time defined by (9·7), makes it possible to use the same set of equations (5·6) as for uniform diffusivity. The only difference is that the time variation of surface temperature *on the*

T-scale is not known before the solution is carried out; the surface temperature is given as a function of true time, t, and the relation between the reduced time T and the true time t has to be constructed from (9·7) as the solution proceeds. This, however, introduces no difficulty either in the mechanical or numerical solution of the equations (5·6).

When the diffusivity does not vary much through the thickness of the material, it may be an adequate approximation to take, at each moment, the value corresponding to the *surface* temperature, so that

$$T = (n^2/l^2)\int D(\theta_s)\,dt. \tag{9·8}$$

Since the surface temperature is given as a function of time, this has the advantage that the distortion of the time scale is known before the solution of the equation (5·6) is started, and the surface temperature is then a known function of the reduced time T. Then there is no difference at all between the treatment of equations (5·6) on the distorted time scale defined by (9·8) and the treatment of those equations for a material of constant diffusivity.

Similar arguments apply, of course, to the treatment of heat flow in solids of other geometrical shapes.

One treatment of two-dimensional heat flow in a cylinder, without the restriction to axial symmetry, has been given in § 3 (*d*). The approximation of taking the diffusivity as uniform throughout the material at each instant, but varying with the time, introduces the possibility of an alternative treatment, which leads to simpler equations than the set (3·15).

The azimuthal variation of the temperature at any time can be expanded in a Fourier series in ϕ:

$$\theta = \theta^{(0)}+\theta^{(1)}\cos\phi+\theta^{(2)}\cos 2\phi+\ldots+\theta^{(-1)}\sin\phi+\theta^{(-2)}\sin 2\phi+\ldots, \tag{9·9}$$

where the coefficients $\theta^{(n)}$ are functions of r and t. Substitution in (9·4) leads to the following set of equations for these functions:

$$\frac{\partial\theta^{(n)}}{\partial t} = D(\bar{\theta})\left[\frac{\partial^2\theta^{(n)}}{\partial r^2}+\frac{1}{r}\frac{\partial\theta^{(n)}}{\partial r}-\frac{n^2}{r^2}\theta^{(n)}\right], \tag{9·10}$$

and, in particular, the axially symmetrical contribution $\theta^{(0)}$ is given by the solution of

$$\frac{\partial\theta^{(0)}}{\partial t} = D(\bar{\theta})\left[\frac{\partial^2\theta^{(0)}}{\partial r^2}+\frac{1}{r}\frac{\partial\theta^{(0)}}{\partial r}\right], \tag{9·11}$$

which is the same as the equation for radial heat flow.

The surface values of the quantities $\theta^{(n)}$ can be determined, as functions of t, from the Fourier analyses of the azimuthal variation of the surface temperature at different times. If the diffusivity is not only uniform but constant, these equations are independent and can be solved separately, and their solutions then added together to give the resultant temperature distribution.

When the variation of diffusivity with mean temperature is taken into account, the quantities $\theta^{(n)}$ are no longer strictly independent, since, as will be seen shortly, the coefficient $D(\bar{\theta})$ entering into the equation for $\theta^{(n)}$ depends on the function $\theta^{(0)}$, but this does not introduce any practical difficulty into the evaluation of a solution of these equations. The mean, over the cross-section of the cylinder of any of the contributions to (9·9) other than the axially symmetrical contribution $\theta^{(0)}$, is zero, hence the mean temperature $\bar{\theta}$ depends only on this contribution $\theta^{(0)}$. So the solution of (9·11) can be carried out first, as for radial heat flow, and the solution determines the mean temperature $\overline{\theta}$ as a function of time. Then for the solution of the other equations of the set (9·10) $D(\overline{\theta})$ is a known function of time.

The r-derivatives in equations (9·10) can be dealt with as in the axially symmetrical case, and this gives

$$\frac{\partial\theta_j^{(n)}}{\partial t}=\frac{D(\overline{\theta})}{2j^2(\delta r)^2}\left[j(2j+1)\,\theta_{j+1}^{(n)}-(4j^2+2n^2)\,\theta_j^{(n)}+j(2j-1)\,\theta_{j-1}^{(n)}\right]. \tag{9·12}$$

The value of $\theta_0^{(n)}$ (for $n\neq 0$) is necessarily zero, and for $n\neq 0$ this value takes the place of equation (3·8) for $n=0$ for the time variation of the central value of the contribution $\theta^{(0)}$.

10. Treatment of the heat flow in the neighbourhood of the transformation temperature

There are several ways of dealing with the abnormal behaviour of the thermal properties in the neighbourhood of the transformation temperature. The simplest and crudest is to take equation (9·5), which assumes the diffusivity to be uniform through the material, or the equivalent equations (9·6) and (9·7), and use them even through the transformation range. At first sight this looks extremely crude, and so it is if the temperature distribution through the steel *while it is going through the transformation range* is required. But trials show that for rates of heating occurring in practice, the *subsequent* behaviour of the internal temperature, for a given time variation of the surface temperature, is only slightly affected by the abnormal thermal properties in the transformation range. This may seem surprising, but an explanation will be found in § 12. Hence, unless the behaviour of the temperature through the transformation range itself is required, it may be an adequate approximation to use (9·5).

A better approximation is provided by (3·4), and a still better one by (9·2) or (9·3). One or other of the latter sets of equations should be used if any details of the temperature distribution through the transition range are required.

Another method is to regard the transformation as taking place at a constant temperature, with absorption or evolution of a latent heat. For this purpose a smooth nominal specific heat curve through the transformation range (say 710–760° C for steel no. 5, for example*) is drawn, and the contribution from this 'specific heat' to the change of total heat H through this range is found by integration. The difference between this and the whole change of total heat is regarded as a latent heat occurring at a definite temperature of transformation (715° C for steel no. 5, see table 41, *Second Report of the Alloy Steels Research Committee* (1939), p. 226). It is convenient to express this latent heat as an equivalent temperature change. If σ_L' is the nominal specific heat at the adopted transformation temperature θ_L, (L/σ_L') is the temperature change which would be produced, in the absence of the transformation, by the latent heat if the specific heat had the constant value σ_L'. This virtual temperature change, which will be written $\Delta\theta_L$, can be regarded as a measure of the thermal effect of the transformation; its value is of the order of 50° C for steel 5, the exact value depending on the way in which the nominal specific heat curve is drawn.

In a substance with a latent heat of transformation at a definite temperature, the transformation at any moment is taking place (if at all) on a surface of the material, not throughout

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* For the specific heat curve, see *Second Report of the Alloy Steels Research Committee* (1939), section IX (3), figures 156 A, B and tables.

a volume.* The temperature gradient (or rather $K(\partial\theta/\partial x)$ if the thermal conductivity is different for the two forms of the substance involved in the transformation) is discontinuous at the surface on which the transformation is taking place, and this surface moves through the substance with a finite velocity. In mechanical integration of the equations, it seems difficult to deal accurately with this situation without a substantial alteration in the machine set-up, which it would be inconvenient to make during the course of a run. But in this temperature region it is an approximation even to take the thermal properties as functions of the temperature only, as already emphasized, and consequently a rougher treatment of the resulting equations can be tolerated here than in other parts of the temperature range. A convenient method has been developed; it looks decidedly rough at first sight, but the results of using it agree closely with those obtained using the better approximation (9·2), except through the transformation range itself, and it has been used extensively in the present work.

This method is based on a crude presentation of Schmidt's method [§ 1 (iii)] in which the material was conceived as built up of laminae in each of which the temperature is uniform. Then in equation (3·4) the left-hand side represents the rate of change of temperature of one of these laminae, and the right-hand side the balance between the heat passing into and out of it. If now a transformation is occurring in one of the laminae, say in that between $x=(j\pm\frac{1}{2})\,\delta x$, the temperature of this lamina is conceived as remaining constant at θ_L until the whole of the lamina has passed through the transformation. If f is the fraction of the material of the layer which has undergone the transformation, this can be represented by replacing (3·4) by

$$\frac{L}{\sigma'_L}\frac{\partial f}{\partial t}=D(\theta_L)\,\frac{\theta_{j+1}-2\theta_L+\theta_{j-1}}{(\delta x)^2}, \tag{10·1}$$

when θ_j reaches θ_L, and keeping θ_j constant at θ_L while the value of f obtained by integrating this equation increases from 0 to 1, that is to say, while the integral of the right-hand side increases by $L/\sigma'_L=\Delta\theta_L$.

This treatment should give a fair approximation to the net heat transfer between this lamina and the adjacent material while this lamina is passing through the transformation range, though details will not be correct. As already mentioned, it has been found that it gives results in close agreement with those obtained by the better approximation (9·2), except at temperatures quite near θ_L, and it has the advantage of being considerably easier to put into effect in the solution of the equations by the differential analyser (see § 11).

In the case of a material such as steel there is reason to suppose that the transformation proceeds at a finite rate depending both on the temperature and on the extent to which the change has already occurred. Let f be the fraction of the material, in the neighbourhood of a point of it, which is in the state characteristic of temperatures above the change point. Instead of taking the heat content H as a function of the temperature θ only through the transformation range, it is probably better to regard it as a function of f and θ. Then

$$\frac{\partial H}{\partial t}=\sigma\frac{\partial\theta}{\partial t}+L\frac{\partial f}{\partial t},$$

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* This can be seen as follows. If the transformation takes place at a definite temperature, then, if it were taking place throughout a volume, the temperature would be constant throughout that volume, there would be no temperature gradient and so no flow of heat through it, and the transformation could not continue.

where $\sigma = \partial H/\partial\theta$ can be regarded as the specific heat of the material, and $L = \partial H/\partial f$ the latent heat associated with the transformation. The heat-conduction equation then becomes*

$$\rho\left(\sigma\frac{\partial\theta}{\partial t}+L\frac{\partial f}{\partial t}\right) = \mathrm{div}\,(K\,\mathrm{grad}\,\theta), \tag{10·2}$$

or, if the variation of the conductivity is neglected,

$$\rho\left(\sigma\frac{\partial\theta}{\partial t}+L\frac{\partial f}{\partial t}\right) = K\nabla^2\theta. \tag{10·3}$$

If adequate quantitative information were available regarding the variation of the specific heat, and the rate $\partial f/\partial t$ at which the transformation proceeds, with θ and f, it would be possible to extend the methods of the present paper to the solution of equation (10·3), and the results should give a theoretical treatment of the variation of the change point with rate of heating or cooling, the isothermal transformation curves ('S-curves'), etc.; and similar data through the solidification range would enable the solidification process to be treated. But at present it does not seem that such data exist. Indeed, at various points of the present work progress has been held up by lack of adequate quantitative data, and whereas before this work was started it could fairly be said that the information available on thermal properties was in advance of the methods of using it, now the methods are ahead of the determination of data required to enable full advantage to be taken of them.

11. Application of the differential analyser to the equation of heat flow with variable diffusivity

Of the various methods in § 9 for handling the equation of heat flow when the thermal properties vary with temperature, the simplest from the point of view of the differential analyser is that which regards the diffusivity as uniform at each instant, with a value appropriate to some mean temperature $\bar{\theta}$. By use of the distorted τ scale of time defined by (9·6), the equations are reduced to the same form as those for constant diffusivity in ordinary time, and the only new feature to be catered for is the relation between the τ-scale of time and true time; this has to be introduced in some way, since the variation of surface temperature is given as a function of true time.

If the diffusivity varies only slightly through the material, it may even be an adequate approximation to take its value at the surface temperature. Then the relation between time measures on the t-scale and on the τ-scale can be evaluated before the solution is begun, and the surface temperature plotted as a function of time on the τ-scale, or more conveniently on the T-scale defined by (9·8), and then there is no difference at all between the procedure with varying diffusivity and with constant diffusivity.

But in using the differential analyser it is almost as easy to adopt a better approximation and to use the diffusivity at some mean temperature $\bar{\theta}$. The choice of a suitable mean temperature will be considered shortly.

It is then most convenient to use the measure of time on the distorted T-scale defined by (9·7) as independent variable for the integration of the heat-flow equations and to derive

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* *Note added in proof, April* 1946. This equation is similar to the equation of heat conduction in a substance in which a chemical reaction is taking place, which has been studied, by the method of the present paper and otherwise, by Crank and Nicolson (1946).

from it the measure of time on the t-scale in terms of which the variation of surface temperature is given.

The distortion of the T-scale relative to the t-scale (or vice versa) can be represented mechanically by connecting the shafts of the machine representing t and T by a continuously variable gear giving a gear ratio proportional to the instantaneous value of D (or $1/D$). An integrator is just such a continuously variable gear; in fact from the definition (9·7) of T it follows that

$$t = \int [l^2/n^2 D(\bar{\theta})]\, dT. \tag{11·1}$$

Hence if an integrator is displaced by an amount proportional to the instantaneous value of $1/D(\bar{\theta})$, and rotated from the shaft whose rotation measures T, its output will be a measure of t, and can be used to drive an input table carrying a graph of the surface temperature θ_s as a function of the true time t. The displacement proportional to $1/D(\bar{\theta})$ is furnished to the machine from a second input table on which a curve of $1/D(\theta)$ as a function of θ is placed, and which is driven by a shaft whose rotation measures the mean temperature $\bar{\theta}$.

It is most convenient to take $\bar{\theta}$ as the temperature at one of the points $x_j = jl/n$ (for the slab) or $r_j = ja/n$ (for the cylinder) which bound the intervals into which the thickness of the slab, or radius of the cylinder, is divided. For steady heating the temperature distribution is approximately parabolic in x (for the slab) or in r (for the cylinder), and for such a distribution the mean temperature is the temperature at $x/l = 1/\surd 3$ for the slab and $r/a = 1/\surd 2$ for the cylinder. This suggests that the best convenient approximation to take for $\bar{\theta}$ is $\theta(\frac{1}{2}l)$ or $\theta(2l/3)$ for the slab, and $\theta(\frac{1}{2}a)$ or $\theta(2a/3)$ for the cylinder, according as a two-interval or three-interval solution is being used. Alternatively, the mean temperature might be taken as $\bar{\theta} = \frac{1}{3}(2\theta_0 + \theta_s)$, which could be constructed on the machine by an adding unit and appropriate gears. The use of a uniform diffusivity is an approximation anyway, and there is no justification for great refinement in the choice of a mean temperature to use; all that is needed is a guide to a better value than the surface temperature.

In this method of handling the heat-flow equations, only two more units of the machine are used than in the treatment of the equations with constant diffusivity (for the same value of n), namely, the input table giving the relation between θ and $1/D(\theta)$, and the integrator which constructs the relation between T and t.

Another method is to take the equations in the form (3·4) and (3·5) with the appropriate value of $D(\theta_j)$ for each x_j. This requires two integrators instead of one for each of the points X_j, since each θ_j is constructed as the integral of the product of two *varying* factors $D(\theta_j)$ and $(\theta_{j+1} - 2\theta_j + \theta_{j-1})$ on the right-hand side of (3·4). In addition to the input table carrying the curve of the surface temperature against time (on the t-scale), it requires an input table, carrying the curve of $D(\theta)$ against θ, for *each* x_j. The curves carried by these tables are all the same, but in running the solution on the machine one has to supply the machine continuously with the values of $D(\theta)$ for several values of θ *simultaneously*, and this cannot be done from a single input table.

Another method is to take the equations in the form (9·2) or (9·3). If equations (9·3) are used, for example, the results of integrating them are the quantities $Q_j = Q(\theta_j)$ where $Q = \rho \int (\sigma/K)\, d\theta$ (see (8·7)), and from these it is necessary to determine the temperatures θ_j

in order to form the integrands for the integration of the equations (9·3). This is done by evaluating Q as a function of θ, as defined by (8·7) from the data on the thermal properties of the material, and using the results to draw a set of graphs of θ as a function of Q (not vice versa), from which each θ_j can be determined from the corresponding Q_j evaluated by the machine. For each x_j this requires only *one* integrator (instead of two as required by equations (3·4)), and also one input table carrying a graph of θ against $Q(\theta)$. Such an input is required for each x_j, although all these input curves are the same, for the reason explained in the previous paragraph. An input table giving θ_s as function of t is also wanted. This method is perhaps the best for use in the neighbourhood of the transformation temperature; in this temperature range, the function Q increases very rapidly with θ; hence the slope of the input curve of θ as a function Q is small, and the temperature is well determined by the value of Q.

Similarly, if equations (9·2) are used, Θ and $H(\Theta)$ would be calculated as functions of θ and the results used in the form of a set of graphs of Θ as a function of $H(\Theta)$. A set of input tables would be required, each carrying one of these graphs, also an input table giving Θ_s as a function of t.

A simpler treatment of the transformation range can be given if the heat absorbed or evolved in this range, over and above that expressed by a smooth nominal specific heat, is regarded as a latent heat L absorbed or evolved at a fixed transformation temperature θ_L. Then equations (3·4) give each $\partial\theta_j/\partial t$ unless $\theta_j = \theta_L$, and for the point (or points) x_j at which $\theta_j = \theta_L$, the relevant equation is (10·1), and θ_j remains constant while f_j changes from 0 to 1. This situation can be represented on the machine by altering the connexions on the output of the integrator which normally evaluates θ_j by integration of $\partial\theta_j/\partial t$. When θ_j reaches θ_L, this output is disconnected from the shaft whose rotation measures θ_j, and this shaft is locked; the integrator output is taken instead to a counter. The machine is then run until the counter reads a value corresponding to $f = 1$; this value corresponds to a temperature change $\Delta\theta_L = L/\sigma'_L$. Then the transformation is complete for that x_j, the normal connexions of that integrator are restored, and the solution continued. This procedure has to be carried out with each integrator in turn as its output reaches the value corresponding to the transition temperature.

Use of equations (9·7) and (5·6), appropriate to a uniform D, with a smoothed nominal specific heat through the transformation range, and of a latent heat at a fixed transformation temperature treated as just described, probably provides the simplest way of treating the heat flow through the transition-temperature range. This method has the advantage of requiring only two input tables, one for $1/D$ as a function of θ, and one for θ_s as a function of t.

Equations (9·3) should give a better approximation, especially if the heat of transformation is regarded as absorbed or evolved over a small range of temperature and not at a single temperature, but this method requires one input table for each x_j as well as one for the time variation of surface temperature.

Probably the best practical method, if personnel to man the required input tables is available, is to use equations (9·7) and (5·6) so long as all the steel is below about 680° C or above 750° C, and (9·3) or (9·2) while the temperature of any part of the steel is between 680 and 750°. The results of § 7 and of § 14 (ii), table 3, suggest that a three-interval solution should give adequate accuracy for practical purposes.

In dealing with surface conditions other than given time variations of surface temperatures, the procedure is rather different according as equation (4·5) or (4·8) (for the slab, or corresponding equations for the cylinder) is used to handle these surface conditions. It will be assumed that the relation between the surface heat flux F and the surface temperature θ_s is given and independent of the time, as this will probably be the most usual case; it is also one of the simplest to handle, and the procedure does not then depend on the form of the relation between F and θ_s. On the other hand, if F does depend explicitly on the time, the procedure will depend on the form of the relation between F, t and θ_s, and needs special consideration for each case.

If the treatment is based on equation (4·5), this has to be regarded as an equation for θ_s, given θ_{n-1} and θ_{n-2}, which are obtained by integrating the approximate equations of the set (3·4). Provided the surface heat flux F depends only on the surface temperature θ_s, as assumed, the left-hand side of (4·5) is a function of θ_s only. The easiest way to obtain the solution of (4·5) for θ_s, continuously as θ_{n-1} and θ_{n-2} are determined by the machine by integration of the appropriate equations for them, is to calculate

$$[2\delta x F(\theta_s)/K(\theta_s)+3\theta_s]$$

as a function of θ_s, and to use the results in the form of a graph of θ_s as a function of

$$[2\delta x F(\theta_s)/K(\theta_s)+3\theta_s]$$

on an input table, which will then furnish θ_s as a function of $4\theta_{n-1}-\theta_{n-2}$.

The alternative treatment using equation (4·8) for the slab or (4·12) or (4·13) for the cylinder, increases the number of differential equations to be solved simultaneously by one for each surface but should give a better approximation as pointed out in § 4.

12. Alternative means of evaluating solutions of the approximate equations

Although the differential analyser provides the quickest and least laborious method of evaluating solutions of the approximate forms of the heat-conduction equation with variable diffusivity, it is in many cases quite practicable to obtain at least a fairly accurate solution without use of it.

The results discussed in § 7 suggest that solutions of accuracy adequate for practical purposes may often be obtained by the use of quite a small number n of intervals to cover the space range, and this is confirmed by some results which will be given in § 14. When this number of space intervals is small, it has been found quite practicable to use numerical methods for the integration of the set of equations (3·4) or (9·3) (for the slab, and the similar equations for radial heating of a cylinder), using an integration formula correct at least to the first difference of the integrand for the time integration, and so substantially more accurate than Schmidt's method, and also using the diffusivity appropriate to the instantaneous temperature at each point.

In particular, when use of one interval from the surface to the central plane of a symmetrically heated slab, or from the surface to centre of a symmetrically heated cylinder, gives adequate accuracy, the relevant equation is (3·5) or (3·8) respectively, or the corresponding finite-difference form of (8·4), (8·6), or (8·8). Then, provided the time variation of surface temperature is smooth, numerical integration of either of these equations, using integration formulae correct to second differences for the integration with respect to time, is simple and quick. The condition of smoothness of the time variation of surface temperature is important, since otherwise the time differences of $\partial\theta_0/\partial t$ are irregular and unsatisfactory for use in integration formulae; this is a practical restriction on numerical methods which does not arise in the use of the differential analyser, which carries out the time integration continuously, instead of step by step.

With $n = 2$, a similar process is still quite practicable in these cases, and does not involve the use of a diffusivity uniform through the material at each instant. But with higher values of n, numerical integration of these sets of equations, even using integration formulae correct only to first differences for the time integration, becomes lengthy and tedious; the number of time steps required to cover a given period is proportional to n^2, and the work involved in each time step is roughly proportional to n at least, so the total amount of work increases as n^3 at least.

For two-dimensional heat conduction in a cylinder with a given, not axially symmetrical, time variation of surface temperature, it has been found practicable to evaluate a solution of (3·15), or rather of the corresponding finite difference form of (8·8), using two intervals $\delta r = \frac{1}{2}a$ in radius and four of length $\delta\phi = \frac{1}{2}\pi$ in azimuth, for which there are five equations in all.

Another method for obtaining quickly the approximate time variation of the central temperature in a slab or cylinder, symmetrically and steadily heated, from the time variation of the surface temperature is given in § 13.

Still another method is suggested by the formal equivalence of the set of equations (3·4) or the set (3·7) to the equations for the potentials in a 'ladder network' of C, R loops, with the resistances in series and capacities in parallel. This equivalence shows that the distribution and time variation of temperature satisfying these equations can be simulated (perhaps on a different time scale) by the time variation of the potential distribution in a corresponding network with suitably controlled potentials applied at points corresponding to the surfaces. This forms the basis of the electrical instrument of Paschkis & Baker (1941, 1942); an experimental instrument of this kind was devised and constructed independently in the course of the present work [see Jackson and others (1944), § 4]. Such an instrument in its present form only applies directly to the evaluation of heat flow in a material whose thermal properties are independent of temperature. However, to the approximation involved in taking the diffusivity uniform throughout the material at each instant, but varying with the time according to the mean temperature $\bar{\theta}$ (see § 9), this instrument can be applied to evaluate heat flow in a material with thermal properties varying with the temperature. The time of operation of the electrical instrument is then proportional to the measure of time on the distorted τ-scale, defined by (9·6), for the corresponding heat flow; the surface temperature would be given as a function of true time t, so a variable-speed gear would be necessary to give the relation between τ and t.

13. The 'time-lag' approximate solution for slow steady heating

Another type of approximate solution of the equation of heat conduction, for a symmetrically heated slab or cylinder, and one whose evaluation requires the minimum of numerical work, is suggested by a property of the exact solution for the case of constant diffusivity, and for surface temperature uniform and increasing at a constant rate. This property, which has been pointed out by Williamson & Adams (1919; see p. 108), is that under these conditions the temperature at any internal point ultimately lags behind the surface temperature by an amount which, when expressed as a time lag, is independent of the rate of rise of surface temperature.

This property was only established by Williamson & Adams for simple geometrical shapes, and as a deduction from the formal solutions which can be obtained for these shapes. But it is more general, and applies to any shape of solid which is bounded in the direction of the flow of heat, whose surface temperature is uniform and increasing at a constant rate. This can be seen as follows.

Let $\chi(\mathbf{r})$ be a function of position satisfying

$$\nabla^2\chi = -1 \tag{13·1}$$

throughout the volume of the solid, and $\chi = 0$ on its surface or surfaces, so that χ is positive throughout it. Then

$$\theta = \beta[t-\chi(\mathbf{r})/D] \tag{13·2}$$

is a solution of (1·1) (for constant diffusivity D) with $\theta = \beta t$ on the surface. Further, the general solution of (1·1) with this surface condition differs from (13·2) by a quantity which is a solution of (1·1) which is zero over the surface; this represents a starting transient which depends on the initial temperature distribution and tends to zero as the time increases (for examples, see Williamson & Adams (1919), formulae (15), (16), (17)). Hence (13·2) represents the ultimate behaviour of the temperature through the solid. The quantity $\chi(\mathbf{r})/D$ is the lag between the times at which the temperature at the surface and the temperature at the point $\mathbf{r}$ pass through any given value; this is independent of the rate of rise of surface temperature, since the equation and boundary conditions determining χ do not involve β.

For a symmetrically heated slab of thickness $2l$, the time lag between surface and centre is $l^2/2D$, and for a symmetrically heated solid cylinder of radius a it is $a^2/4D$; these values follow directly from formulae (15) and (16) of Williamson & Adams (1919). The numbers 2 and 4 in the formulae for the time lag in these two cases are very closely related to the coefficients 2 and 4 in equations (3·5) and (3·8) respectively. In fact, in both these cases the distribution of temperature through the slab or cylinder ultimately become parabolic (see formulae (15) and (16) of Williamson & Adams 1919), and then the fundamental approximations (3·2) of the present method is exact, so equations (3·5) and (3·8) are exact and can be used to go from centre to surface in one interval δx for the plane, and similarly for the cylinder; and the time-lag property for constant rate of rise of surface temperature can easily be deduced from these equations.

Now for a substance for which the diffusivity varies with the temperature, it has been shown in § 9 that if the diffusivity does not vary greatly through the material at any one time (though it may vary considerably in the course of time), the heat flow can be reduced approximately to the heat flow in a substance of constant diffusivity on a distorted scale of time which has been called the τ-scale.

Hence, if the rise of surface temperature is constant *on the τ-scale*, the central temperature will follow the surface temperature with a time lag which is constant *on the τ-scale* (and which on this scale has a value of $\frac{1}{2}$ for a slab and $\frac{1}{4}$ for a cylinder), and which varies on the t-scale according to the value of the diffusivity, that is, according to the mean temperature.

Further, consider a variation of surface temperature at a constant rate on the τ-scale, followed by another variation at a different constant rate (see figure 1). It takes about four times the time lag for the starting transient to become negligible, so that the central temperature follows the surface temperature after the lapse of the appropriate time lag. Hence, if each straight portion of the curve of θ_s, against τ, lasts longer than a period of about 2 in τ for the slab, or 1 in τ for the cylinder, the curve of central temperature against time will consist of a series of nearly straight portions parallel to successive portions of the curve of surface temperature, displaced horizontally from it by the appropriate time lag, and joined together by curved portions in the neighbourhood of the sudden changes of rate of rise of surface temperature.

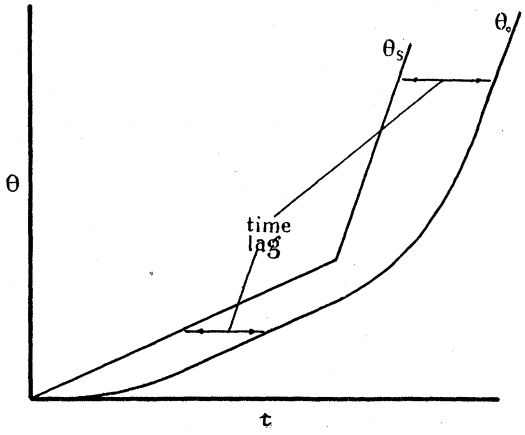

FIGURE 1

Now suppose these sudden changes smoothed out, then it seems likely that so long as $\partial\theta_s/\partial t$ *changes slowly*, the central temperature will follow the surface temperature approximately with a time lag in τ of $\frac{1}{2}$ for a slab or $\frac{1}{4}$ for a cylinder. It is not necessary that $\partial\theta_s/\partial t$ should be *small* in order that this approximation should apply; what is important is that it should not change by a substantial amount in 1 or 2 units of time on the τ-scale.

In terms of true time t, this means that so long as $[1/D(\bar{\theta})]\,\partial\theta_s/\partial t$ changes *slowly*, the central temperature will follow the surface temperature with a time lag $l^2/2D(\bar{\theta})$ for a slab, or $a^2/4D(\bar{\theta})$ for a cylinder. The variation in $[1/D(\bar{\theta})]\,\partial\theta_s/\partial t$ may arise from the variation of either factor in this product; if the heating is *steady* the second factor varies slowly, and if the heating is also *slow*, the first factor varies slowly, so this 'time-lag' solution may be expected to apply in cases of slow, steady heating.

Between the times $t_s(\theta)$, $t_0(\theta)$ at which surface and centre respectively pass through any temperature θ, the mean temperature $\bar{\theta}$ passes through this temperature, so that the time-lag solution can be written

$$t_0(\theta) = t_s(\theta) + l^2/2D(\theta) \tag{13·3}$$

for a symmetrically heated plate or slab of thickness $2l$, and

$$t_0(\theta) = t_s(\theta) + a^2/4D(\theta) \tag{13·4}$$

for a symmetrically heated cylinder of radius a.

This approximate solution is so simple that it would be surprising if it is not already known, but we have been unable to find any reference to it in the literature.

In most heat treatments other than quenching, the variation of surface temperature is slow, and often very slow, on the τ-scale of time, which is the natural scale of time to use in heat-flow contexts and in which the unit of time is l^2/D for a slab and a^2/D for a cylinder, so the 'time-lag' solution may often give results adequate for practical purposes. An example will be found in § 14.

The slowness, in this sense, in the variation of surface temperature is the main reason why the internal temperature distribution after all the steel has passed through the transformation range does not depend on the details of the heat flow through that range, and why, therefore, a crude treatment of the thermal properties through that range is adequate provided that details of the temperature distribution through that range are not required.

More generally, if the surface temperature is given as a function of the time, the temperature distribution in the material at any time depends to a considerable extent only on the behaviour of the surface temperature over an immediately previous period of about a unit of time on the τ-scale; earlier stages in the thermal history have little influence on the temperature distribution at that time or subsequently.

14. Some examples and comparison with experimental results

Some examples will now be given of the use of the method of this paper for the approximate solution of the equation of heat conduction in a material whose diffusivity is a function of temperature. In three of these examples, some experimental data on internal temperatures are available for a comparison with the results of calculation. In the fourth, the purpose of the calculation was the determination of the internal temperature distribution in a case taken from works practice, for which the only available measurements were of the surface temperature.

Comparison between calculated and observed temperature distributions do not always show complete agreement, and sometimes the differences are considerable (reference is made to one such case in (ii) below). In discussing such comparisons, two separate possible sources of discrepancy must be kept clearly distinct. Firstly, calculations have to be done for definite assumed conditions (for example, axial symmetry may be assumed in the case of a bar or ingot), and for assumed values for the thermal properties, and these may not provide adequate representations of the condition and values occurring in the experiments; for example, the heating may depart appreciably from axial symmetry, or the transformation may be so delayed that thermal properties in the neighbourhood of the change point may not be those assumed. Secondly, although the form of the equation of heat conduction used, and the values for the thermal properties used in it, may provide an adequate representation of the actual conditions, approximations are involved in reducing it to the finite-difference form to which the method of solution is applicable.

The small number of intervals into which the space range is divided in evaluating these solutions might suggest, at first sight, that the approximations involved in the use of finite differences in the space co-ordinates might well be the main source of discrepancy between calculated and observed internal temperature distributions. However, this source is not serious in practice. Moreover, although an exact solution of the equations of heat conduction with variable diffusivity cannot be obtained, an estimate can be made of the errors arising from the use of the analytical approximations (3·2) and (3·3) on which the method of solution is based, and corrections made for them if appreciable; for an example, see (ii) below. Although formally this correction is itself only approximate, it is probable that the residual errors after it has been applied will usually be small, and then the only source of discrepancies outside the limits of experimental error is the difference between the conditions, including values and thermal properties, for which the solution of the equation of heat conduction was calculated and the condition of the experiment.

(i) 14-*ton ingot*. The first trial of the method of calculation with diffusivity varying with temperature, and the first comparison between the results of such a calculation and experiment, was carried out on the results of such an experiment on a 14-ton octagonal ingot of mean diameter 28 in., the data being immediately available. This experiment had been carried out in the Research Department of Messrs Hadfields Ltd. in 1927, as one of the first steps in the study of the internal temperature distribution in masses of steel under conditions of works practice.

The ingot was a medium carbon steel of composition resembling steel no. 5 (*Second Report of the Alloy Steels Research Committee*, 1939, table 46). It was heated in a furnace, and measurements of temperature made by thermocouples. Difficulties were experienced due to couple breakage; the most satisfactory results are shown in graphical form in figure 2.

For the calculation of the central temperature, the heat flow was considered to be radial; the values of the thermal properties were taken to be those of steel no. 5; the radius was divided into six equal intervals for the solution of equations (3·7) and (3·8), and the solution was carried out by the differential analyser.

For this preliminary work the approximation expressed by (9·5) was adopted; in this the diffusivity is taken as uniform through the material at each instant though varying with the time. Further, the latent heat at the change point was neglected.

It had previously been verified that slight irregularities in the curve of surface temperature against time had little effect on the central temperature, so the surface temperature curve obtained from the observations was smoothed and the smooth curve adopted as input for the evaluation of a solution. The smoothed curve of surface temperature, and the calculated curve of central temperature, against time, are shown in figure 2, for comparison with the experimental results.

In view of the difficulty of making measurements of this kind under works conditions, and the limitations of the pyrometric methods available in 1927, the agreement is satisfactory, and was good enough to encourage further work both on the experimental side and in the further development of the method of calculating heat flow. Indeed, in view of the difficulties found in later work on the laboratory scale, the degree of agreement of these early experimental results with the results of calculation appears surprisingly good.

(ii) *Laboratory measurements of* $1\frac{3}{4}$ *and* $3\frac{1}{2}$ *in. bars*. Experimental conditions can be more closely controlled in experiments on the laboratory scale than those on the works scale, and a set of experiments was carried out in the research laboratories of Messrs Hadfields Ltd. on the heating of $1\frac{3}{4}$ and $3\frac{1}{2}$ in. diameter bars under conditions which were designed to give a close approach to radial heating; the experimental methods are given in the paper by Jackson and others (1944). For each experiment, the time variation of surface temperature

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was used as input for a solution of the set of equations (3·7) and (3·8), and the calculated time variation of central temperature was compared with that observed.

The first experiment, on a $1\frac{3}{4}$ in. bar, gave results which showed no agreement at all between the observed and calculated central temperatures; the calculated central temperature agreed closely with that given by the 'time-lag' solution of § 13, whereas the time lag between surface and observed central temperature was about four times as large as that given by (13·5). The experiments had, however, been carried out with considerable care, and apart from this comparison with calculation the results appeared quite satisfactory and would have been accepted without question had the calculation not been made.

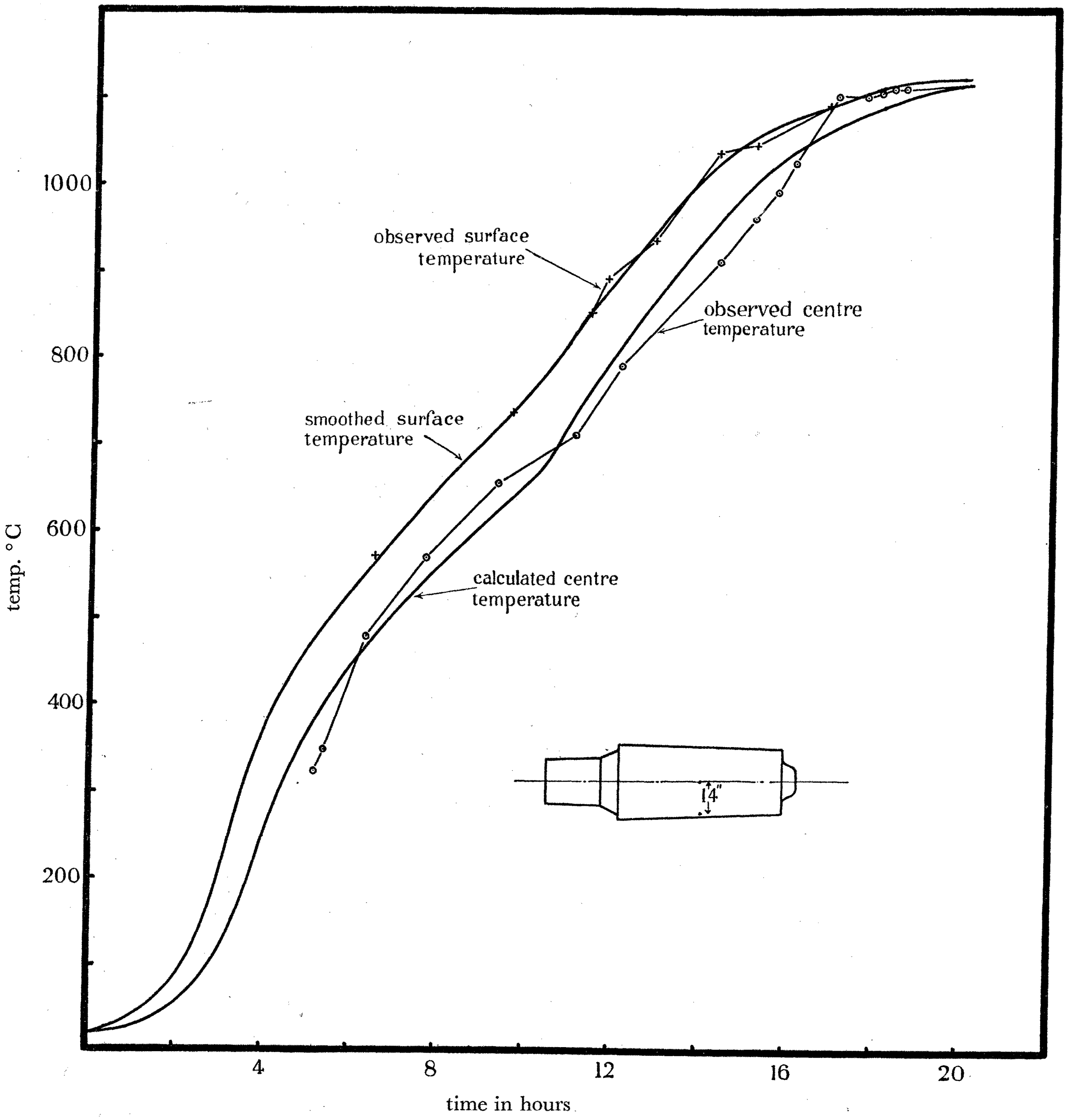

FIGURE 2. Time variation of surface temperature, and of observed and calculated central temperatures, in heating of 14-ton ingot.

The violent discrepancy between calculation and observation found in this experiment emphasized the difficulties in making satisfactory experimental determinations of the temperature distribution in the interior of masses of steel under non-steady conditions, and the care which must be exercised in accepting such experimental determinations without at least an approximate check from calculation of an appropriate solution of the equation of the heat conduction. It led to a critical examination and revision of the experimental technique involved, which finally resulted in satisfactory agreements between observed and calculated central temperatures. For a discussion of the experimental aspects of this work, see the paper by Jackson and others (1944).

The heating of the $1\frac{3}{4}$ in. bar was rather inconveniently fast for the satisfactory recording of the temperatures, and the best experiments, both on this account and for the better approximation to radial heating, were those done on a $3\frac{1}{2}$ in. bar. This was of steel no. 5; it was normalized before the experiment, in which it was heated to 960° C, cooled in air to 430° C, and then again heated.

Results for the first heating are given in table 3 in the form of observed surface and central temperatures, and four calculated central temperatures, as functions of time. The four calculated central temperatures were obtained as follows:

(*a*) by solution of equation (3·8) (replaced by the corresponding finite difference form of (8·8) through the range 680–750° C), taking one interval $\delta r = a$ from the surface to the centre of the cylinder;

(*b*) similarly, but taking two intervals $\delta r = a/2$ from surface to centre;

(*c*) by use of solutions (*a*) and (*b*) to correct approximately for the errors due to the use of the approximations (3·2) and (3·3) and so to obtain a better approximation to the solution of the partial differential equation;

(*d*) by the 'time-lag' solution (13·5).

The calculated solutions (*a*) and (*b*) were obtained by numerical integration rather than by the differential analyser, so that the small differences between these solutions should be certainly significant and not subject to possible doubts regarding the accuracy of the differential analyser with graphical input. The two-interval solution was not carried over the whole period, as it was apparent that over considerable ranges its difference from the one-interval solution would be less than 1° C.

The leading terms in the errors due to the use of approximations (3·2) and (3·3) are proportional to $1/n^2$, where n is the number of intervals into which the space range is divided; hence the solution of the partial differential equation lies outside the interval between the solutions (*a*) and (*b*) with $n = 1$ and 2 respectively, at a distance from the latter of about one-third the difference between them. In the present case the greatest difference between the two solutions is only 3° C, so even including the transformation range the greatest error in the two-interval solution is only about 1° C and that of the one-interval solution only about 4° C. The approximation made in obtaining the latter solution appears at first sight to be very crude, and the accuracy of the results given by it seems remarkable.

The accuracy of the time-lag solution (13·5) over the range for which it is applicable (that is, excluding the transformation range 680–750° C) is also surprisingly good. This suggests that, provided the measurement of internal temperatures is thoroughly reliable, the time lag between surface and central temperatures passing through the same value might be used as a means of measuring thermal properties under transient conditions. Further reference will be made to this possibility later.

Solution (*c*) provides an estimate of the solution of the partial differential equation, with the adopted values for the diffusivity and for the time variation of surface temperature, with a probable error of less than 1° C, and any greater departures of the observed central temperature from that so calculated must be due to departures of the thermal properties or the variation of surface temperature adopted in the calculations from those actually applicable to the material and conditions of the experiment, or to errors in measurement of the central temperature.

TABLE 3. HEATING OF $3\frac{1}{2}$ IN. BAR. COMPARISON OF CENTRAL TEMPERATURES AS CALCULATED BY DIFFERENT METHODS, AND AS OBSERVED. STEEL NO. 5

time t min.	surface temperature θ_s (° C)	central temperature θ_0 (° C), calculated				
		a	*b*	*c*	*d*	observed
0	25	25	25	25	25	25
1	107	$67\frac{1}{2}$	65	64	58	—
2	170	131	$130\frac{1}{2}$	$130\frac{1}{2}$	130	114
3	216	$183\frac{1}{2}$	$183\frac{1}{2}$	$183\frac{1}{2}$	$184\frac{1}{2}$	—
4	255	$226\frac{1}{2}$	$226\frac{1}{2}$	$226\frac{1}{2}$	$226\frac{1}{2}$	210
$5\frac{1}{2}$	312	283	—	—	282	271
7	368	$337\frac{1}{2}$	—	—	$336\frac{1}{2}$	329
$8\frac{1}{2}$	421	$389\frac{1}{2}$	—	—	389	383
10	468	$437\frac{1}{2}$	—	—	$436\frac{1}{2}$	433
$11\frac{1}{2}$	512	481	—	—	480	477
13	553	$521\frac{1}{2}$	—	—	$520\frac{1}{2}$	518
$14\frac{1}{2}$	587	558	—	—	558	556
16	619	$590\frac{1}{2}$	—	—	$589\frac{1}{2}$	586
$17\frac{1}{2}$	647	619	—	—	619	617
19	672	$645\frac{1}{2}$	—	—	646	644
$20\frac{1}{2}$	693	$667\frac{1}{2}$	—	—	666	664
22	712	686	$685\frac{1}{2}$	$685\frac{1}{2}$	683	686
$23\frac{1}{2}$	729	702	701	701	699	704
25	740	712	$710\frac{1}{2}$	710	—	720
$26\frac{1}{2}$	742	717	714	713	—	725
28	746	724	$723\frac{1}{2}$	$723\frac{1}{2}$	—	726
$29\frac{1}{2}$	752	734	—	—	$742\frac{1}{2}$	728
31	761	743	—	—	747	732
34	785	$764\frac{1}{2}$	—	—	765	754
37	816	796	—	—	799	794
40	844	$829\frac{1}{2}$	—	—	830	825
43	864	$853\frac{1}{2}$	—	—	854	—
46	882	873	—	—	874	874

Calculated values of θ_0:

(*a*) obtained by use of equation (3·8) with one interval $\delta r = a$ from surface to centre;

(*b*) obtained by use of equations (3·7) and (3·8) with two intervals $\delta r = \frac{1}{2}a$ from surface to centre;

(*c*) obtained by combination of results under (*a*) and (*b*) to correct (approximately) for the errors of the finite-difference approximations (3·2) and (3·3);

(*d*) obtained by use of the 'time-lag' method (§ 13).

The general agreement between the observed central temperature and that so calculated is very satisfactory, considering the difficulties of the experimental measurements. It is certainly good enough for outstanding substantial differences between observed and calculated results to be regarded as significant and not spurious. The main outstanding difference is in the behaviour of the temperature in the range 700–770° C, in which the observed curve shows a flattening at a higher temperature than does the calculated curve. This flattening is due to the absorption of heat at the α–γ transformation which takes place in this range. The values of the thermal properties used in the calculation were taken from the *Second Report of the Alloy Steels Research Committee* (1939), and refer to very slow heating, when for this steel the transformation takes place in a small temperature range in the neighbourhood of 715° C. It is clear that under the conditions of the experiment the transformation is delayed and occurs mainly at about 725° C. This delay is still more marked in the cooling, in which the transformation takes place at about 680° C.

Similar calculations for the second heating showed appreciably less good agreement with experiment than those of the first, to a degree which suggested that the thermal properties had been appreciably modified by the heating to 960° and subsequent cooling. This can be best examined by comparing the time lags between surface and centre temperatures in the two cases, since, as already pointed out in § 13, this time lag is independent of the rate of heating and, although the time-lag solution is not exact, errors are likely to be similar so long as the time variations of the surface temperature are similar, so that alterations in the time lag can be regarded as significant indications of changes in thermal properties.

Values of the time lag in the two heatings and in the cooling, at a series of temperatures, are given in table 4, and appear to confirm the suggested alteration of the thermal properties. They do not show whether the alteration is in the specific heat or conductivity or both, but the results show the possibility of using the time lag to obtain at least partial information on the variation of thermal properties in the course of a heat treatment.

TABLE 4. $3\frac{1}{2}$ IN. DIAMETER BAR. OBSERVED TIME LAG BETWEEN SURFACE AND CENTRE

	time-lag in minutes		
temp. ° C	first heating	cooling	second heating
100	1·0	—	—
200	1·1	—	—
300	1·1	—	—
400	1·2	—	—
500	1·4	1·7	1·7
550	1·5	1·6	1·7
600	1·6	1·8	2·0
650	1·9	change point	2·1
700	2·1	2·1	2·5
750	change point	2·0	change point
800	2·5	2·0	2·5
850	2·0	1·8	1·9

(iii) *Asymmetrical heat flow in a cylinder.* In some later experiments under works conditions, an ingot of 11 in. square section, of nickel-chromium-molybdenum steel of composition similar to steel no. 12 (Alloy Steels Research Committee 1939), was heated in a works furnace, and measurements of temperature at a section half way along the length of the ingot were made by thermocouples arranged as shown in figure 3. The observed surface temperature showed marked departures from uniformity round the perimeter of the section, and this case seemed an admirable one for exploratory work in the treatment of heat conduction without the restriction to axial symmetry.

For reasons which will be explained shortly, the calculations were carried out for a *cylinder* of diameter $2a$ equal to the side of the square section. The method of partial heat flows indicated at the end of § 9 was used until the highest temperature reached 680° C. From there two separate calculations were made by different methods to examine the difference between their results; in one calculation the method of partial heat flows was continued right through the transformation temperature range; in the other the finite difference form of (8·8), in plane polar co-ordinates, corresponding to the finite difference form (3·15) of (3·14), was used, with two intervals $\delta r = \frac{1}{2}a$ in radius and four intervals $\delta\phi = \frac{1}{2}\pi$ in azimuth. The differential analyser being engaged at the time on other work, the calculations were carried out numerically and were found to be quite practicable.

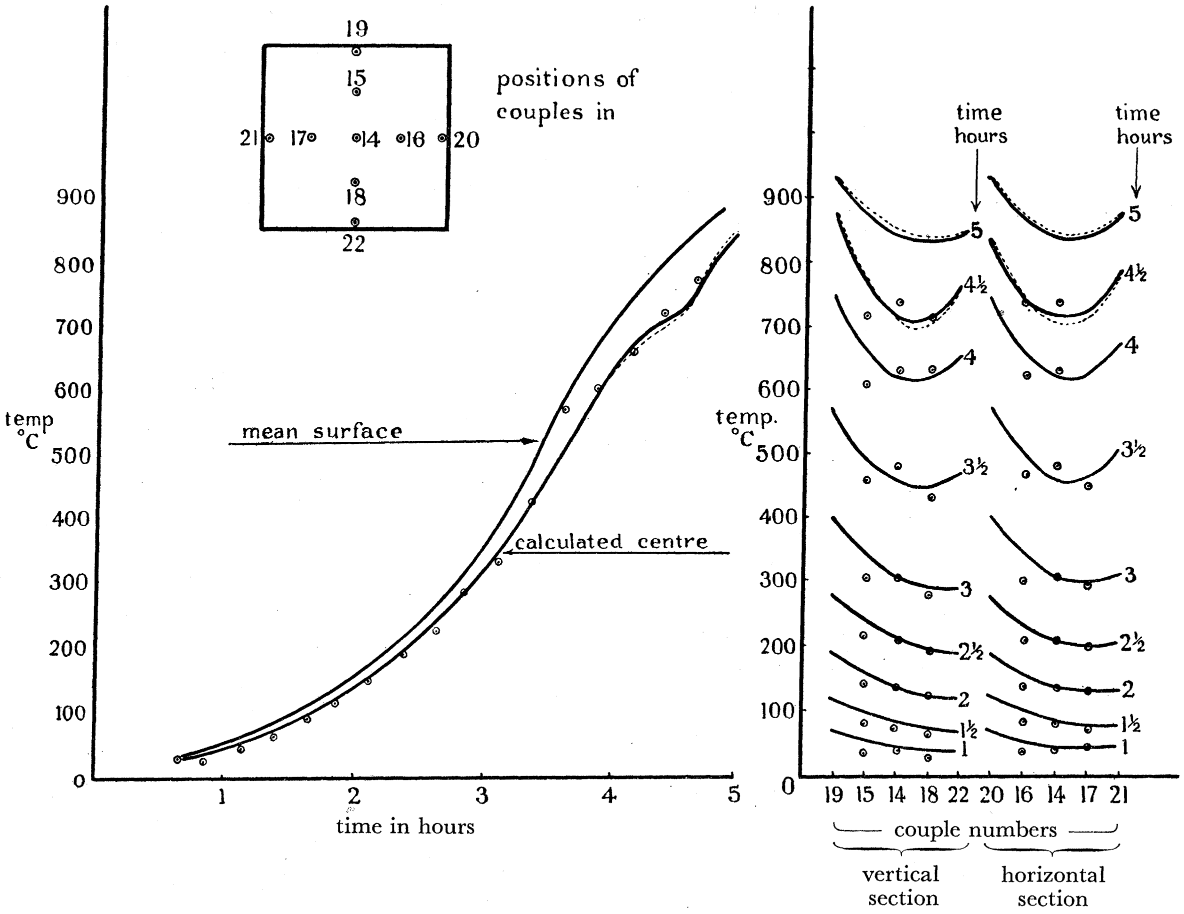

FIGURE 3. Asymmetrical heating of 11 in. square ingot. Left, curves show time variation of mean observed surface temperature and calculated central temperature; points show observed central temperature. Right, curves show calculated variations of temperature in horizontal and vertical sections at half-hourly intervals; points show observed temperatures.

This treatment of the heat flow means that the solution refers to the interior of the cylinder inscribed in the square prism formed by the ingot itself. This would, of course, be carried out without reference to the surface temperature distribution of the ingot, if the temperature distribution on the circumference of the section of the cylinder were known. In general, this would require temperature measurements in the interior of the ingot, but so long as only four intervals are used to cover the range 2π of azimuth, the temperatures at the points of contact of the cylinder with the square prism in which it is inscribed provide all that is required for the approximate solution of the equation of heat conduction for the cylinder. If, as in the present case, the temperatures at these four points form all the available data on the distribution of surface temperature, it seems better to solve the equation of heat conduction for the inscribed cylinder than for the whole square prism. For with the equations appropriate to a square prism, the best that could be done would be to take a net of side a, which would give only a single equation, namely, that for the central temperature, and, moreover, this equation is the same as that for a cylinder treated with one interval $\delta r = a$ from surface to centre; whereas by using the equations appropriate to asymmetrical heat flow in the inscribed cylinder, it is possible to take smaller intervals in the radius, and to treat more closely the departure from radial flow of heat.

The distribution of temperature given by the observations suggested, before the calculation had been started, that the observed values were not all reliable, but it was impossible to determine from the measurements alone which were at fault. Under these circumstances it is arbitrary which are accepted; but since for the purposes of calculation the values of the surface temperature are the essential quantities, it was assumed that these were correct, and the calculations were carried out on this basis. But its arbitrary nature must be remembered in comparing the results for the internal temperatures with those observed.

The results are shown in figure 3. The left-hand part of this figure shows by curves the time variation of the mean of the four recorded surface temperatures, and the calculated central temperature; the observed values of the central temperature are shown by plotted points. In the right-hand half of figure 3, the curves show the calculated variations of temperature along the horizontal and vertical diameters at various times; the plotted points indicated measured temperatures. In the higher part of the temperature range, the full curves show the solution calculated by the method of partial heat flows (see end of § 9) and the broken curve that calculated by use of the finite difference form of (8·8). The latter should be considerably the more accurate, as the approximation on which the method of partial heat flows is based fails through the transformation temperature range.

The instantaneous temperature distributions calculated by the method of partial heat flow are nearly parabolic; those calculated by the more accurate method depart appreciably from the parabolic form (see particularly the vertical distribution at time $4\frac{1}{2}$ hr.), as would be expected when there is a large variation of specific heat over the cross-section. The way in which the results calculated by the rough partial heat flow method quickly approach those calculated by the more accurate method as soon as all the material has passed through the transformation temperature range should be noted. This is typical of the way in which quite rough methods, too rough to be used through the transformation range itself, have often been found in the course of this work to give good results again quite soon after the transformation range has been passed.

The agreement between observed and calculated internal temperatures is fair. The calculated variation of temperature along both diameters is considerably more regular than that observed, and this strengthens the suggestion that not all the temperature measurements are reliable. As there is no evidence, apart from comparison with the results of calculation, as to which are to be accepted as reliable, there seems no object in discussing this comparison in detail. The main result of this work is that it establishes the practical possibility of carrying through calculations for asymmetrical two-dimensional flow in a cylinder.

(iv) *Large forging.* As an example of the application of the method to the calculation of the internal temperature distribution in masses of steel under conditions of works practice, figure 4 shows the results of calculation for a large ingot in heating, forging and reheating operations. The surface temperature was measured by optical pyrometers, and was assumed uniform for the purpose of the calculation, which was done on the differential analyser, using three intervals in the radius of the cylinder. For simplicity, the radius during the forging period was taken as constant and equal to the mean of its actual initial and final values. The heat generated during forging was neglected, as, although its total amount could be estimated, its distribution was unknown; also neglect of it leads to an under-estimate of the internal temperatures after forging, and so to an over-estimate of the time required for reheating which is safer in practice than an underestimate. If the distribution of the heat generated internally were known, it could be included in the solution of the equations.

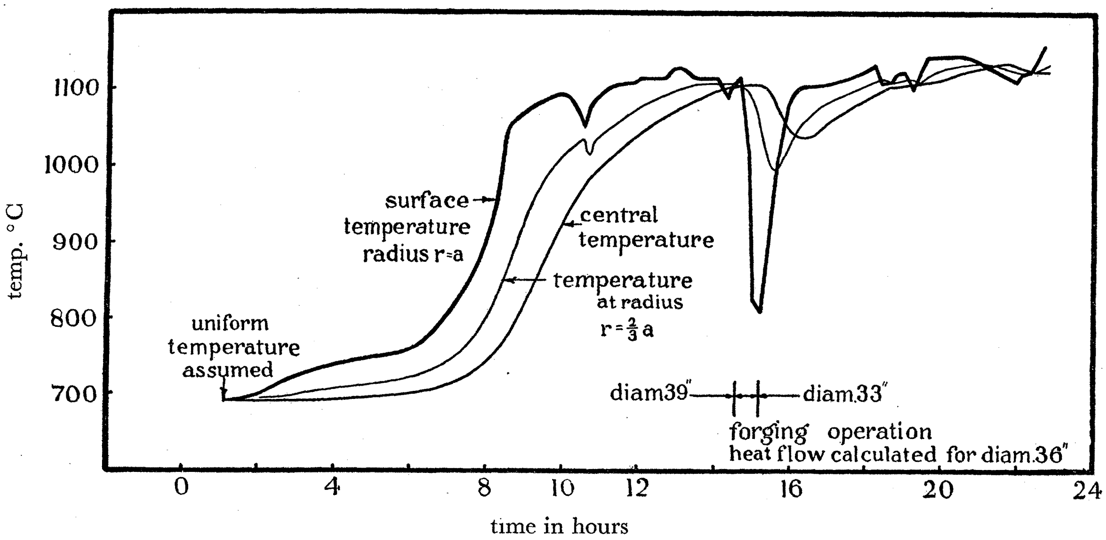

FIGURE 4. Time variation of temperature distribution during heating and forging of a 39 in. diameter steel ingot. Surface temperature observed; internal temperatures calculated.

These examples will serve to give an idea of the range and scope of this method for evaluating solutions of the equation of heat conduction without restriction to constant diffusivity, and of the application of the differential analyser as a practical means of carrying out these calculations. Although the examples refer to heat conduction in steel, the method is clearly applicable as it stands to heat conduction in other materials and to diffusion. Moreover, only minor extensions are required to handle inhomogeneous bodies of simple geometrical forms, such as a cylinder surrounded by a sheath, of uniform thickness, of another material, the diffusivities of both materials depending on temperature. Another minor extension from the point of view of technique is to heat conduction in materials in which heat is being generated; some examples of results have already been given elsewhere (Hartree 1943).

PART IV. COOLING OF A CAST INGOT

15. EQUATIONS OF HEAT FLOW INCLUDING RADIATIVE HEAT TRANSFER, AND THEIR APPROXIMATE TREATMENT

In the casting of an ingot, molten steel is poured into a mould. Soon after the outer surface of the steel has become solid, a gap forms between the outside surface of the ingot and the inner surface of the mould, and increases in width as the mould heats up and expands, and the ingot cools and contracts. From the time when the gap is formed, which is a very short time after pouring compared with the whole time during which the ingot is cooling, transfer of heat from the cooling ingot takes place by conduction through the ingot and through the mould walls, and by radiation across the gap.

It seemed that it might be possible to extend methods of the previous sections to deal with this situation. There were available results of some experiments in which the time variations of the temperatures of the mould at its outer surface, half way through its wall, and just within its inner surface,* and also the central temperature of the ingot had been measured; the outer surface temperature would provide data for a solution of the appropriate equations, and the observed central temperature would serve for comparison with the results of such a solution.

In these experiments, the ingot and mould were of square section with rounded corners. For a first experimental application of this technique for solution of the heat flow equations, it seemed an adequate approximation to treat the ingot as a circular cylinder of area equal to that of the actual square section, and the mould walls as portions of uniformly heated planes, and the following discussion will refer to this case. The cast iron of the mould and the steel of the ingot may not have the same thermal properties. Where it is necessary to distinguish between them, suffixes i and m will be used to indicate values referring to the ingot and to the mould respectively; values of the thermal properties for each must, of course, be taken at the appropriate temperature.

Let a be the radius of the ingot, l the thickness of the mould walls, $\theta_0, \theta_1, \ldots \theta_n$ the temperatures at points at equal spacings $\delta r = a/n$ along a radius of the cylinder (θ_0 referring to the centre and θ_n to the surface), and $\theta_{n+1} \ldots, \theta_N$ the temperatures at points at equal spacings $\delta x = l/(n-N)$ (not necessarily equal to δr) through the mould wall, θ_{n+1} referring to the inner surface and θ_N to the outer surface; $\theta_N = \theta_s$ will be regarded as given as a function of the time, so that there are N unknowns θ_0 to θ_{N-1}, and N equations are therefore required to determine them.

The transfer of heat by conduction through the ingot and through the mould wall can be treated exactly as already described in Part II; this provides n differential equations for the internal temperatures θ_0 to θ_{n-1} in the ingot, and $N-n-2$ for the internal temperatures θ_{n+2} to θ_{N-1} in the mould. Two more are required to give the temperatures on either side of

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* The measured temperatures on the inside surface of the mould were regarded as not altogether reliable. The work described in this Part arose originally from an enquiry as to what information could be obtained from the measurements of temperature on the outside of the mould, half way through it and at the centre of the ingot only. It will appear that internal temperatures are only required at one time, in order to provide initial conditions for the work, and that thereafter the time variation of temperature on the outside surface of the mould alone provides sufficient data for the determination of the internal temperature distribution and its variation with time.

the gap between ingot and mould. These are given by the equations of radiative heat transfer across the gap. Fortunately these do not depend on the width of the gap, so that the fact that little is known quantitatively about the width of this gap is no barrier to the solution of the heat flow equations.

Radiative heat transfer between two parallel plane surfaces has been considered by Saunders (1929), Fishenden & Saunders (1932) and others, for general values of the emissivities of the two surfaces. If η_i, η_m are the emissivities of the surfaces of the ingot and mould respectively, and α is Stefan's constant, the net radiative heat flux from ingot to mould is

$$\frac{\eta_m\eta_i}{\eta_m+\eta_i-\eta_m\eta_i}\,\alpha(\theta_n^4-\theta_{n+1}^4). \tag{15·1}$$

If the two emissivities are the same and equal to η this becomes

$$[\eta/(2-\eta)]\,\alpha(\theta_n^4-\theta_{n+1}^4), \tag{15·2}$$

but even if the emissivities of the two surfaces are not equal, (15·2) and (15·1) are equivalent if

$$\eta=2\eta_m\eta_i/(\eta_m+\eta_i), \tag{15·3}$$

that is, η is the harmonic mean of the emissivities of the two surfaces. With this understanding, (15·2) can be used for the net radiative heat flux, without implying that the surfaces have the same emissivity.

This radiative heat flux across the gap must be equal to the heat flux by conduction immediately inside the materials of the ingot and mould, which are $-K_i(\partial\theta/\partial r)_n$ and $-K_m(\partial\theta/\partial x)_{n+1}$ respectively. Hence

$$-K_i(\partial\theta/\partial r)_n=-K_m(\partial\theta/\partial x)_{n+1}=[\eta/(2-\eta)]\,\alpha(\theta_n^4-\theta_{n+1}^4). \tag{15·4}$$

This provides the two further equations which are necessary to enable the N temperatures θ_0 to θ_{N-1} to be determined.

The temperature gradient at the surfaces of the solids which occur in (15·4) can be dealt with by any of the methods indicated in § 4, and thus the equation of radiative transfer of heat can be brought within the scope of the methods already considered.

If the approximation (4·3) is used for these temperature gradients, the first of equations (15·4) becomes just a linear algebraic equation

$$(K_i/\delta r)\,(3\theta_n-4\theta_{n-1}+\theta_{n-2})=(K_m/\delta x)\,(3\theta_{n+1}-4\theta_{n+2}+\theta_{n+3}). \tag{15·5}$$

16. Application of the differential analyser

In the application of the differential analyser, the only new feature is the treatment of the equations (15·4) expressing the relations involved in the radiative heat transfer across the gap between ingot and mould. The temperature up to θ_{n-1} in the ingot and from θ_{n+2} outwards in the mould are given by integration of the corresponding time derivatives given by (3·7) and (3·8) for the ingot and (3·4) for the mould.

If (4·3) is used for the surface temperature gradient, (15·4) becomes a pair of simultaneous algebraic equations for θ_n, θ_{n+1} in terms of the neighbouring temperatures. This situation can be handled on the machine by an interconnexion of two input tables with the output shafts from the integrators evaluating the neighbouring temperatures, as shown schematically in figure 5. This interconnexion determines θ_n and θ_{n+1}, which are then combined with the neighbouring temperatures to form $\partial\theta_{n-1}/\partial t$ and $\partial\theta_{n+2}/\partial t$. The set-up shown in figure 5 is only one of several possibilities but it seems to be the most convenient; in this set-up the fourth power is taken of the *lower* temperature θ_{n+1}, and the *higher* temperature θ_n is found as the fourth root of its fourth power; these two steps could be reversed, but the arrangement shown in figure 5 is preferable.

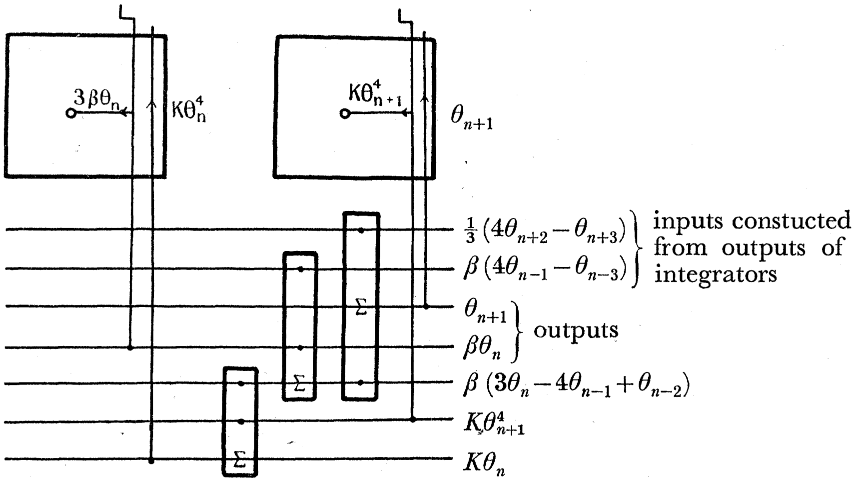

Figure 5. Schematic diagram of differential analyser connexions for handling the equations of radiative heat transfer in the form

$$\beta(3\theta_n-4\theta_{n-1}+\theta_{n-2})=\theta_{n+1}-\tfrac{1}{3}(4\theta_{n+2}-\theta_{n+3})=-k(\theta_n^4-\theta_{n+1}^4),\quad \left(\beta=\frac{1}{3}\frac{K_i}{K_m}\frac{\delta x}{\delta r},\quad k=\frac{1}{3}\frac{\eta}{2-\eta}\frac{\alpha\delta x}{K_m}\right).$$

The input quantities to this portion of the machine set-up are $\frac{1}{3}(4\theta_{n+2}-\theta_{n+3})$ and $\beta(4\theta_{n-1}-\theta_{n-2})$; the output quantities from it are θ_{n+1} and $\beta\theta_n$.

17. Results, comparison with experiment, and discussion

Experimental results were available for ingots of side $2a = 9$, 11 and 12 in. In all cases the temperature of the steel remains above the transformation temperature and that of the mould below it over nearly the whole time covered by the observations. This was convenient for exploratory work on the heat flow equations with radiative transfer, as the major temperature variation of thermal properties, which occurs in the transformation range, is not involved. Accurate values of the thermal properties for the cast iron of the mould were not available over the whole range concerned, and it seemed adequate at this stage to take a constant value D_m for the diffusivity for it. Also for the steel no accurate values of thermal properties were available in the higher part of the temperature range, and consequently a constant value D_i was taken for its diffusivity. The value of the coefficient $\eta/(2-\eta)$ in the radiative transfer equation (15·4) was taken as 0·9, the corresponding value of the mean emissivity η is 0·95 very nearly.

In the solution of the heat flow equations, the thickness of the mould wall was covered in two intervals, and for the first trial solutions the ingot was taken in one interval from centre to surface; this is equivalent to assuming a parabolic temperature distribution through it.

Later, a three-interval approximation was taken for the heat flow in the ingot, though the temperature gradient at the ingot surface was obtained from the one-interval formula (17·1) below. The ingot temperatures were then θ_0 (centre), θ_1, θ_2, θ_3 (surface), and the mould temperatures θ_4 (inner surface), θ_5 and $\theta_6 = \theta_s$ (outer surface).

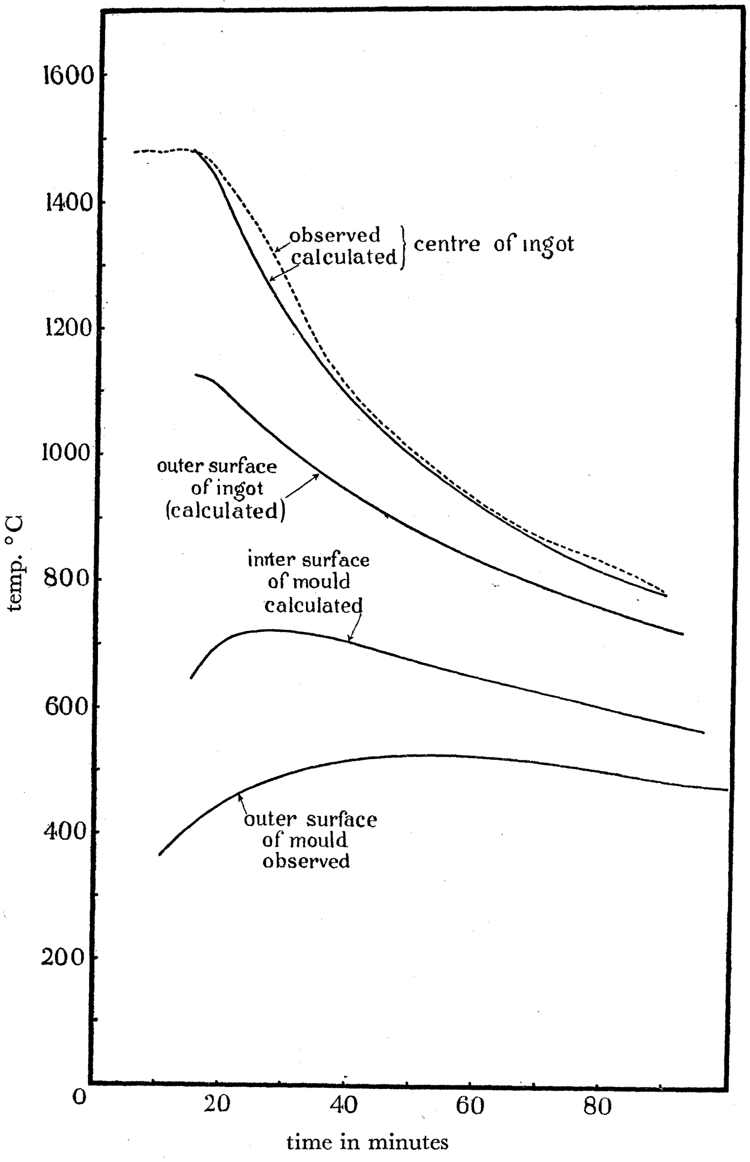

FIGURE 6. Time variation of temperature distribution in 11 in. diameter steel ingot and mould after casting.

The observed temperatures were θ_0, θ_4, θ_5, θ_s, of which θ_4, as already mentioned, was regarded as not reliable. The record of the observed temperatures started from the pouring of the molten steel into the mould. If the latent heat of fusion of steel were known, it would be possible to use the methods of § 10 to start from this stage, or rather from a slightly later stage at which it can be safely assumed that the gap between ingot and mould wall has been formed; this is certainly well before the ingot is solid throughout. But the available values for the latent heat were so diverse that it seemed best to start from a time when all the steel was solidified; this is fairly well marked by the end of the horizontal portion of the curve of central temperature; for the 11 in. ingot (see figure 6), this occurred at about 15 min. from the pouring. In this time, the mould had heated up considerably, and it was necessary somehow to obtain a temperature distribution through the mould as well as through the ingot in order to provide initial conditions for the solution of the heat flow equations. This was done as follows.

For the initial distribution through the ingot, a parabolic distribution was assumed. Then the surface temperature gradient $(\partial\theta/\partial r)_3$ at the surface of the ingot is $2(\theta_3-\theta_0)/a$. For the mould, $(\partial\theta/\partial x)_4 = -(3\theta_4-4\theta_5+\theta_s)/l$ if the approximation (4·3) is used, so the equations of radiative transfer (15·4) become

$$(2K_i/a)\,(\theta_0-\theta_3) = (K_m/l)\,(3\theta_4-4\theta_5+\theta_s) = [\eta/(2-\eta)]\,\alpha(\theta_3^4-\theta_4^4). \qquad (17\cdot1)$$

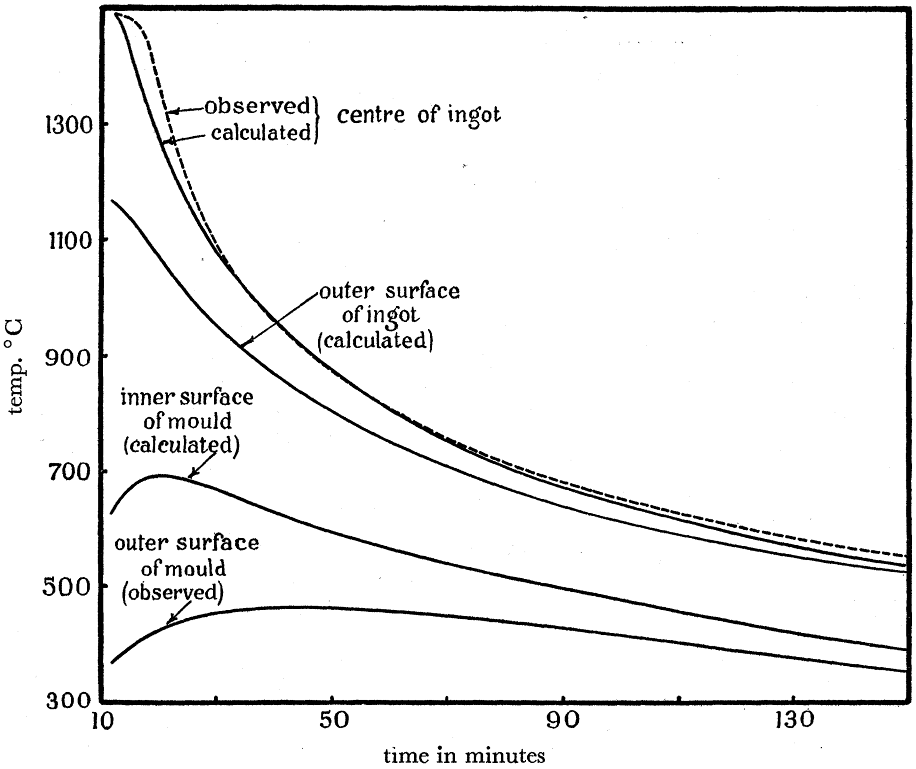

FIGURE 7. Time variation of temperature distribution in 9 in. diameter steel ingot and mould after casting.

If θ_0, θ_5, θ_s are taken from the experimental records, θ_3 and θ_4 can be obtained by solution of these equations, and then from θ_0, θ_3 and the assumed parabolic distribution through the ingot, θ_1 and θ_2 are obtained. It would be expected that the effects of the initial temperature distribution would become negligible after a time whose measure on the τ-scale is a few units, and a^2/D, the unit of time on the τ-scale, is about 4 min. for the 11 in. ingot. This is small compared with the whole period of cooling, so even if the initial temperature distribution thus determined is rather rough, it should not greatly affect the solution after about 10 min.

Figure 6 gives the results for the 11 in. ingot. It shows the observed outer temperature of the mould, the results of integrating the heat flow equations using this time variation of surface temperature and the observed central temperature. Figure 7 gives similar results for the 9 in. ingot.

It must be remembered that the observed central temperature is not used in these solutions of the heat flow equations, except for the single value required to provide the initial conditions for the solution, so that the comparison of observed and calculated temperatures is a real comparison of independently determined quantities. The agreements are quite remarkably good; in fact, considering the approximations made (constant diffusivity for both ingot and mould, for example), one might be inclined to say that it is almost too good.

The results certainly show that it is feasible to evaluate the internal temperature distribution in an ingot cooling in a mould from knowledge of the temperature on the outside surface of the mould only. If the accuracy of the available thermal data, for the steel at high temperatures and for the cast iron of the mould, justified the increase in elaboration of the machine set-up, it would be possible to include the variation of thermal properties both of the steel and of the cast iron with temperature in this work.

The excellence of the general agreement between the calculated and observed central temperatures suggests that differences, where they occur, may be significant. The main difference occurs near the beginning of the calculated solution, which drops sharply away from the melting point, whereas the observed curve has a continuous slope and falls away more gently. This difference may arise from the initial temperature distribution through the ingot differing substantially from the parabolic distribution assumed in the calculations. But it may also arise from solidification taking place not at a definite temperature but over the range between liquidus and solidus in the equilibrium diagram, which for this steel (no. 12 of Table 40 of the *Second Report of the Alloy Steels Research Committee*) is about 20° C; part of the latent heat of fusion would be evolved over this range, and would delay the initial fall of central temperature. To include this in the calculated solution, it would be necessary to know not only the latent heat of fusion, but its contribution to the effective specific heat over this temperature range.

The agreement between observed and calculated temperatures was not nearly so good for the temperatures θ_4, θ_5, at the inner surface of the mould and half way through its wall as for θ_0, the central temperature of the ingot. As already mentioned, at the time the calculations were made, the observed temperature θ_4 at the inner surface of the mould was regarded as not certainly reliable, and no use of it was made in the calculation. On the other hand, the observed values of θ_5, one of which was used to start the calculation, were not originally suspect. However, since the calculations were carried out, a repeat experiment has been made which seems to confirm the original values of θ_4 and to throw doubt on those of θ_5; further, for the 11 in. ingot the observed curves of θ_5 and θ_s against time cross over in the course of the cooling, and it seems most unlikely that this is real, as it would imply an inward flow of heat in the outer part of the mould. This suggests that it might have been better to start from a temperature distribution derived from solution of (17·1) with the observed values of θ_0, θ_4 and θ_s, rather than of θ_0, θ_5 and θ_s, but such a change of initial conditions would probably not affect the solution much after the first 10 min. or so.

Figures 8 and 9 show results, for the central temperature only, obtained from solutions for the 11 in. ingot calculated with different conditions, or with different approximations, to see their effect.

The total range of θ_s is so small (about 100° C) in comparison with that of θ_0 that it seemed worth while examining just how sensitive θ_0 is to variations in θ_s, and whether adequate accuracy for practical purposes could be attained by taking some constant value. Accordingly, three solutions were taken for θ_s constant at 100, 400 and 500°. The results are shown in figure 8. The central temperature θ_0 in each of them differs appreciably from that obtained using the observed θ_s; the result for $\theta_s = 500°$ forms the best approximation, as would be expected since this is nearest to the mean temperature over the period concerned.

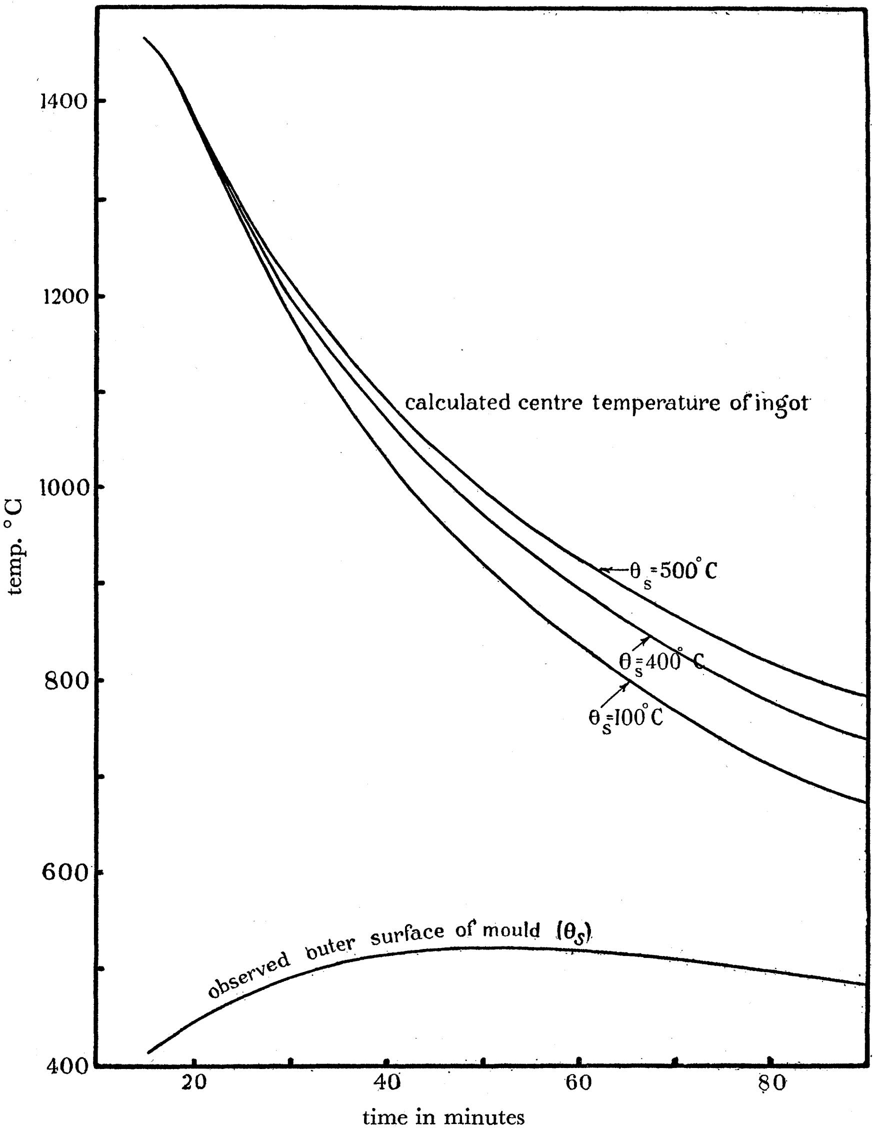

FIGURE 8. Cooling of 11 in. ingot in mould. Effect of temperature of outer surface of mould (θ_s). Central temperature of ingot calculated for θ_s constant at 100, 400 and 500° C.

Figure 9 shows the final result compared with two earlier results; in both of these the heat flow equation in the ingot was taken in one interval instead of in three intervals, and in one $\eta/(2-\eta)$ was also taken as unity instead of 0·9.

The success of this work made it seem worth while trying to deal with the solidification process, despite the uncertainty in the latent heat of fusion. For this purpose, only the 11 in. ingot was considered, and a three-interval solution of equation (10·3) was used, with a latent heat of fusion of 49 cal./g., it being assumed that all the latent heat was evolved at each point before the temperature began to fall. This gave a time of solidification of 20 min. for the 11 in. ingot, which is somewhat larger than the observed time, but is of the right order of magnitude. Further work on this subject awaits the provision of reliable values for the latent heat of steel and the thermal properties at temperatures up to the melting-point.

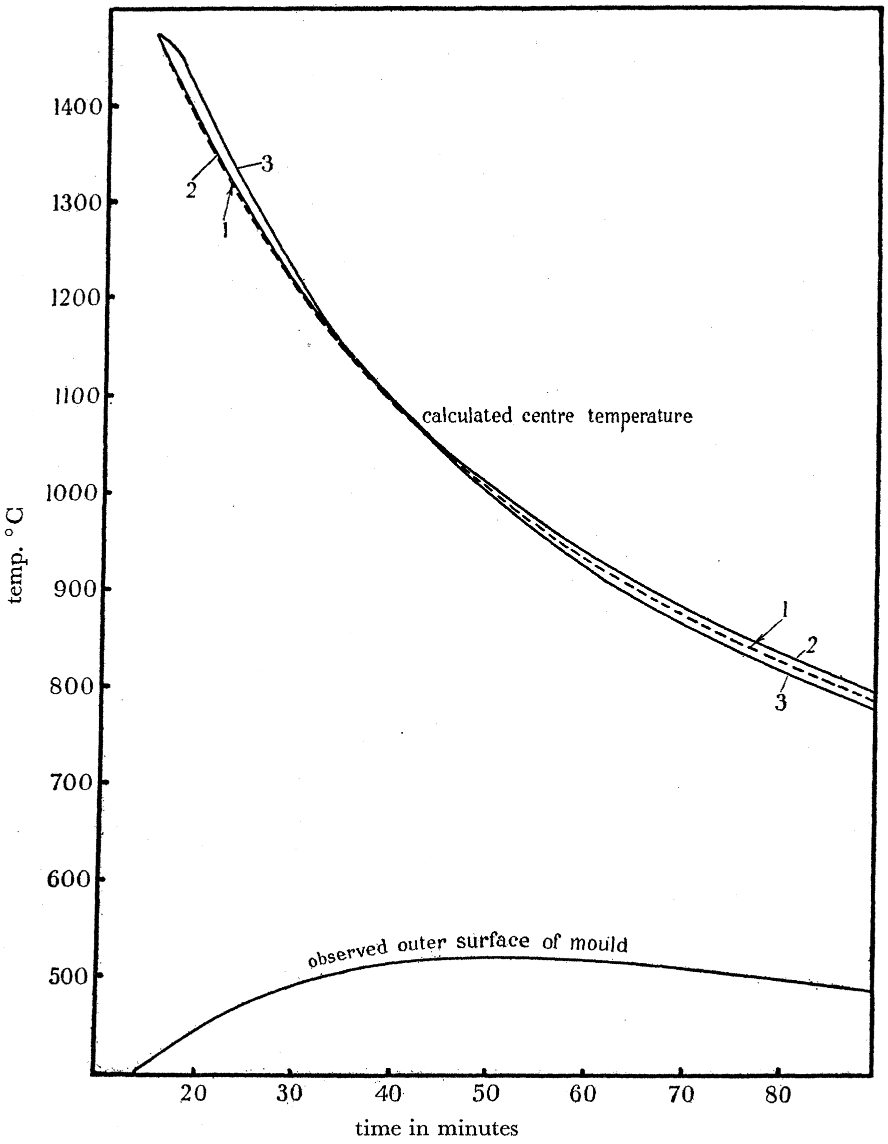

FIGURE 9. Cooling of 11 in. ingot in mould. Effect of method of calculation and of assumed emissivity on calculated central temperature of ingot. Curve (1), one interval in radius of ingot, $\eta/(2-\eta) = 0{\cdot}9$. Curve (2), one interval in radius of ingot, $\eta/(2-\eta) = 1{\cdot}0$. Curve (3), three intervals in radius of ingot, $\eta/(2-\eta) = 0{\cdot}9$.

CONCLUSIONS

The examples given in §§ 14 and 17 give some indication of the power of the method here considered for evaluating solutions of the equation of heat conduction. There is no restriction to constant thermal properties, empirical curves for the time variation of surface temperature can be used, or other, possibly non-linear, types of boundary conditions can be introduced, and the treatment can be extended to include a radiative transfer across a gap in the interior of the solid in which heat conduction is taking place; conduction across the surface of separation of two materials, of different thermal properties, in contact could be handled by a method similar to that used for the treatment of radiative heat transfer. In none of the examples here considered has there been any internal generation of heat, but only a minor extension of the method is needed to include this.

It will be clear, too, that the method is not restricted to the equation of heat conduction. As already explained in § 1, the characteristic feature of the kind of problem to which it can be applied is that the domain over which the solution is required is open in one independent variable (which in practice will often be the time variable, as here, but is not necessarily so); the integration is carried out in this variable, finite differences being used in the other variables. For mechanical integration, equations which are of higher order than the first, in the variable in which integration is being carried out, make so much demand on machine capacity that their treatment needs larger machines than are at present available in this country; and even equations which are first order in this variable, but are not otherwise very simple in structure, require more capacity than is at present available. So the only equations to which the method has been applied are of rather simple form, and first order in time. Even in this restricted field there seems to be considerable scope for the application of the method, which will be greatly extended if substantially greater machine capacity ever becomes available.

This work has been carried out under the auspices of the Thermal Treatment Sub-Committee of the Alloy Steels Research Committee, reporting to the Iron and Steel Industrial Research Council, and the authors also gratefully acknowledge the assistance which has been received from the National Physical Laboratory who have supplied them with much valuable data on the physical properties of steels.

They are also indebted to the Ministry of Supply for permission for their differential analyser group at Manchester University to assist in this work, and to Mr J. Howlett and Mrs P. Nicolson for contributions to the work at various points.

The authors' thanks are also offered to the Directors of Messrs Hadfields, Ltd., Sheffield, in particular to Mr W. J. Dawson, for permission to include in the paper the results of work carried out in the Research Department of the Company, and to Mr T. H. Middleham, Mr D. Knowles and Mr J. D. Cresswell for assistance in connexion with experimental work.

APPENDIX

Analytical solutions of the equations of one-dimensional heat flow for constant diffusivity

As explained in the text (§ 7), analytical solutions both of the exact equation (1·1) of one-dimensional heat flow, and of the set of approximate equations (3·4), (3·5), were required for D constant, to provide tests, firstly of the accuracy of the approximation introduced by the replacement of the exact differential equation by this set of approximate equations, and secondly of the accuracy of the solution of this set of equations obtained by the differential analyser.

It is most convenient for this purpose to take the equations in their dimensionless forms, namely,

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial X^2} \tag{5·3}$$

for the partial differential equation, and

$$\frac{\partial\theta_j}{\partial T} = \theta_{j+1} - 2\theta_j + \theta_{j-1} \tag{5·6}$$

for the set of ordinary differential equations.

The technique of Heaviside operators provides the most convenient analytical tool for the solution of these equations with given surface conditions.

Consider first the partial differential equation (5·3). If p is written for $\partial/\partial\tau$, it becomes

$$\frac{\partial^2\theta}{\partial X^2} = p\theta. \tag{A 1}$$

The solution symmetrical about $X = 0$, and equal to the surface temperature $\theta_s(\tau)$ at $X = 1$, is formally

$$\theta(\tau) = \frac{\cosh p^{\frac{1}{2}}X}{\cosh p^{\frac{1}{2}}}\,\theta_s(\tau) \tag{A 2}$$

and in particular the central temperature is given by

$$\theta_0(\tau) = (\operatorname{sech} p^{\frac{1}{2}})\,\theta_s(\tau). \tag{A 3}$$

As is usual with operational expressions of this type, this formal solution can be expanded in two ways, in partial fractions or in negative exponentials,* and interpretation of these series then gives two alternative expansions of $\theta(\tau)$ as a function of τ, one suitable for evaluation for large τ and the other for small τ.

Expansion in partial fractions

Consider first the expansion in partial fractions of the operator $(\cosh p^{\frac{1}{2}}X/\cosh p^{\frac{1}{2}})$, operating on the surface temperature in (A 2).

The general expansion of $f(p)/F(p)$ in partial fractions is

$$\frac{f(p)}{F(p)} = \frac{f(0)}{F(0)} + \Sigma_k \frac{f(\alpha_k)}{\alpha_k F'(\alpha_k)}\,\frac{p}{p-\alpha_k}, \tag{A 4}$$

where the α_k's are the roots of $F(\alpha) = 0$.

Here $\quad f(\alpha) = \cosh\alpha^{\frac{1}{2}}X, \quad F(\alpha) = \cosh\alpha^{\frac{1}{2}}, \quad \alpha F'(\alpha) = \tfrac{1}{2}\alpha^{\frac{1}{2}}\sinh\alpha^{\frac{1}{2}},$

and the roots α_k of $F(\alpha) = 0$ are given by

$$\alpha_k^{\frac{1}{2}} = (2k+1)\,i\,\tfrac{1}{2}\pi, \quad \text{or} \quad \alpha_k = -[(2k+1)\,\tfrac{1}{2}\pi]^2 \tag{A 5}$$

for integer values of k (including zero; only zero and positive values are relevant as negative values of k do not give distinct roots α_k). Hence

$$\alpha_k F'(\alpha_k) = (-1)^{k+1}\,(2k+1)\,\tfrac{1}{4}\pi$$

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* See, for example, Jeffreys (1927), chapter 5.

and (A 4) gives

$$\frac{\cosh p^{\frac{1}{2}}X}{\cosh p^{\frac{1}{2}}} = 1-\frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\cos(2k+1)\frac{1}{2}\pi X}{2k+1}\frac{p}{p+[(2k+1)\frac{1}{2}\pi]^2}. \tag{A 6}$$

When the operator on the right-hand side of (A 6) is applied to the surface temperature θ_s, it gives a series of terms like

$$\frac{p}{p+[(2k+1)\frac{1}{2}\pi]^2}\theta_s(\tau) \tag{A 7}$$

multiplied by numerical coefficients. Now

$$\frac{p}{p+\beta}\phi(\tau) = \left(1-\frac{\beta}{p+\beta}\right)\phi(\tau),$$

and

$$\left.\begin{aligned}\frac{1}{p+\beta}\phi(\tau) &= e^{-\beta\tau}\int_0^{\tau} e^{\beta\tau'}\,\phi(\tau')\,d\tau'\\ &= \int_0^{\tau} e^{-\beta(\tau-\tau')}\phi(\tau')\,d\tau',\end{aligned}\right\} \tag{A 8}$$

and the use of this enables the various functions (A 7) to be expressed in terms of integrals like

$$\int_0^{\tau} e^{-[(2k+1)\frac{1}{2}\pi]^2(\tau-\tau')}\,\theta_s(\tau')\,d\tau',$$

which can then be evaluated numerically or perhaps analytically.

This procedure is necessary when the surface temperature θ_s is an empirical function of time specified only by a graph or table. But when it is given by an algebraic formula, as in the work considered in § 7, it is often easier to replace θ_s in (A 7) by its operational representation, and then to interpret the whole resulting operational expression.

For example, for a surface temperature given by Heaviside's unit function

$$\begin{aligned}\theta_s(\tau) = H(\tau) &= 0 \quad (\tau<0),\\ &= 1 \quad (\tau>0),\end{aligned}$$

(case I of § 7), (A 7) is just

$$\frac{p}{p+[(2k+1)\frac{1}{2}\pi]^2}H(\tau) = e^{-[(2k+1)\frac{1}{2}\pi]^2\tau},$$

so (A 2), (A 6) give, for this case,

$$\theta(\tau) = 1-\frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\cos(2k+1)\frac{1}{2}\pi X}{(2k+1)}e^{-[(2k+1)\frac{1}{2}\pi]^2\tau}, \tag{A 9}$$

and in particular the central temperature is

$$\theta_0(\tau) = 1-\frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}e^{-[(2k+1)\frac{1}{2}\pi]^2\tau}. \tag{A 10}$$

The surface temperature taken in case II of § 7, namely,

$$\begin{aligned}\theta_s(\tau) &= 0 \quad (\tau<0),\\ &= \beta\tau \quad (\tau>0),\end{aligned}$$

has the operational representation* $\theta_s(\tau) \equiv \beta/p$,
so (A 7) becomes just

$$\frac{1}{p+[(2k+1)\,\frac{1}{2}\pi]^2} \equiv \frac{1}{[(2k+1)\,\frac{1}{2}\pi]^2}\,[1-e^{-\{(2k+1)\frac{1}{2}\pi\}^2\tau}],$$

so that
$$\theta(\tau) = \beta\left[\tau - \frac{16}{\pi^3}\sum_{k=0}^{\infty}(-1)^k\frac{\cos(2k+1)\,\frac{1}{2}\pi X}{(2k+1)^3}\{1-e^{-[(2k+1)\frac{1}{2}\pi]^2\tau}\}\right]. \tag{A 11}$$

And the surface temperature taken in case III of § 7, namely

$$\begin{aligned}\theta_s(\tau) &= 0 && (\tau<0)\\ &= \tfrac{1}{3}[50-e^{-\tau/5}(\tau^2+10\tau+50)] && (\tau>0)\end{aligned}$$

has the operational representation

$$\theta_s(\tau) \equiv \frac{50}{3}\frac{1}{(5p+1)^3}.$$

So
$$\begin{aligned}\frac{p}{p-\alpha_k}\theta_s(\tau) &\equiv \frac{50}{3}\frac{p}{p-\alpha_k}\frac{1}{(5p+1)^3}\\ &= \frac{50}{3}\left[\frac{1}{(5\alpha_k+1)^3}\frac{p}{p-\alpha_k} - \frac{5}{(5\alpha_k+1)}\left\{\frac{p}{(5p+1)^3}+\frac{1}{(5\alpha_k+1)}\frac{p}{(5p+1)^2}+\frac{1}{(5\alpha_k+1)^2}\frac{p}{(5p+1)}\right\}\right]\\ &\equiv \frac{1}{3}\left[\frac{50}{(5\alpha_k+1)^3}e^{\alpha_k\tau} - \frac{e^{-\tau/5}}{(5\alpha_k+1)}\left\{\tau^2+\frac{10\tau}{5\alpha_k+1}+\frac{50}{(5\alpha_k+1)^2}\right\}\right],\end{aligned} \tag{A 12}$$

and $\theta(\tau)$ is made up of a series of functions of τ of this kind, with the different values of α_k given by (A 5), with coefficients functions of X as in (A 6).

Expansion in negative exponentials

For small values of τ, the Fourier series form for the solution of the heat flow equations which is given by the expansion of $(\cosh p^{\frac{1}{2}}X/\cosh p^{\frac{1}{2}})$ in partial fractions is only slowly convergent, and so is inconvenient for numerical evaluation. Then it is convenient to use an alternative expansion of the operational form of the solution, in negative exponentials.

For this purpose the operator $(\cosh p^{\frac{1}{2}}X/\cosh p^{\frac{1}{2}})$ is written (with $q = p^{\frac{1}{2}}$ for convenience)

$$\begin{aligned}\frac{\cosh p^{\frac{1}{2}}X}{\cosh p^{\frac{1}{2}}} = \frac{e^{qX}+e^{-qX}}{e^q+e^{-q}} &= e^{-q(1-X)}\,[1+e^{-2qX}]\,[1+e^{-2q}]^{-1}\\ &= e^{-q(1-X)}-e^{-q(3-X)}+e^{-q(5-X)}- \dots +[e^{-q(1+X)}-e^{-q(3+X)}+e^{-q(5+X)}- \dots],\end{aligned} \tag{A 13}$$

and in particular, for $X=0$ $\quad \operatorname{sech} p^{\frac{1}{2}} = 2[e^{-q}-e^{-3q}+e^{-5q}\dots]$. (A 14)

Now it can be shown that

$$e^{-q\xi}\theta_s(\tau) = \frac{1}{2\sqrt{\pi}}\int_0^{\tau}\theta_s(\tau')\,e^{-\xi^2/4(\tau-\tau')}\,[\xi/(\tau-\tau')^{\frac{3}{2}}]\,d\tau',$$

so that for any time variation of surface temperature θ_s, the results of the operation of each term of (A 13) or (A 14) on θ_s can be evaluated. But, as with the expansion in partial fractions,

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* As usual, an operator written without an operand expressly indicated is to be understood as operating on the unit function $H(\tau)$. The symbol $\equiv$ is used to indicate the relation between a function and its operational representation; that is to say $f(p) \equiv \phi(\tau)$ means $f(p)\,H(\tau) = \phi(\tau)$.

if θ_s is given by an analytical formula it is often simpler to replace this formula by its operational representation, multiply by the operator on the right-hand side of (A 13) or (A 14), and interpret the result term by term.

For example, for case I of § 7 $\qquad \theta_s(\tau) = H(\tau),$

so (A 3), (A 13) give just
$$\theta_0(\tau) = 2[e^{-q} - e^{-3q} + e^{-5q} + \dots]\,H(\tau). \tag{A 15}$$

Now if erfc z is written for the error function complement
$$\operatorname{erfc} z = \frac{2}{\surd\pi}\int_z^\infty e^{-w^2}\,dw,$$

then
$$e^{-q\xi} \equiv e^{-q\xi}H(\tau) = \operatorname{erfc}(\xi/2\tau^{\frac{1}{2}}),$$

so (A 15) becomes
$$\theta_0(\tau) = 2[\operatorname{erfc}(1/2\tau^{\frac{1}{2}}) - \operatorname{erfc}(3/2\tau^{\frac{1}{2}}) + \operatorname{erfc}(5/2\tau^{\frac{1}{2}}) - \dots]. \tag{A 16}$$

Now erfc z tends to zero rapidly as z increases beyond about $z = 2$, so this series converges rapidly for small τ, which is just the region where the Fourier series form (A 9) converges slowly.

Similarly, case II of § 7, $\qquad \theta_s(\tau) \equiv \beta p^{-1},$

so (A 3), (A 13) give
$$\theta_0(\tau) \equiv 2\beta[p^{-1}e^{-q} - p^{-1}e^{-3q} + p^{-1}e^{-5q} - \dots]. \tag{A 17}$$

Now
$$p^{-1}e^{-q\xi} \equiv 4\tau\,\text{ii}\operatorname{erfc}(\xi/2\tau^{\frac{1}{2}})$$

where ii erfc z is the second integral of erfc z, and has been evaluated and tabulated (Hartree 1936) for use in just this context. Hence (A 17) gives, for case II of § 7,
$$\theta_0(\tau) = 8\beta\tau[\text{ii}\operatorname{erfc}(1/2\tau^{\frac{1}{2}}) - \text{ii}\operatorname{erfc}(3/2\tau^{\frac{1}{2}}) + \text{ii}\operatorname{erfc}(5/2\tau^{\frac{1}{2}}) - \dots]. \tag{A 18}$$

Values of $\theta_0(\tau)$ for small values of τ in table 1 have been calculated from (A 16) and (A 18).

The finite difference approximation

Consider now the set of ordinary differential equations (5·6), and let p now stand for d/dT instead of for $\partial/\partial\tau$. Then this set of equations becomes
$$\theta_{j+1} - (2+p)\,\theta_j + \theta_{j-1} = 0.$$

The solution of this recurrence relation which is symmetrical about $j = 0$, and has $\theta = \theta_s$ at $j = n$, is formally
$$\theta_j(T) = \frac{z^j + z^{-j}}{z^n + z^{-n}}\,\theta_s(T), \tag{A 19}$$

where z is such that
$$z^2 - (2+p)\,z + 1 = 0. \tag{A 20}$$

At first sight it might be expected that solution of (A 20) for the substitution of the result in (A 19) would give an operator involving $p^{\frac{1}{2}}$. But it should be noted that despite appearances, the operator in (A 19) is simply the ratio of two polynomials in p.

On substitution of $z = e^{\gamma}$, (A 19) becomes
$$\theta_j(T) = \frac{\cosh j\gamma}{\cosh n\gamma}\,\theta_s(T), \tag{A 21}$$

and (A 20) gives
$$p = 2(\cosh\gamma - 1) = 4\sinh^2\tfrac{1}{2}\gamma. \tag{A 22}$$

From (A 22) $\gamma = \cosh^{-1}(1-\frac{1}{2}p)$ and substitution in (A 21) would express the operator formally as a function of p; but it is more convenient to work in terms of γ.

To deal with this situation an extension of (A 4) is needed. What is required is an expansion in partial fractions, *in terms of p*, of expressions like $g(\gamma)/G(\gamma)$, where $p = \phi(\gamma)$. If γ_0 is such that $\phi(\gamma_0) = 0$, and γ_k are the roots of $G(\gamma) = 0$ (only those roots which give distinct values of $\phi(\gamma)$ being included), the appropriate expansion is

$$\frac{g(\gamma)}{G(\gamma)} = \frac{g(\gamma_0)}{G(\gamma_0)} + \Sigma_k \frac{\phi'(\gamma_k)}{\phi(\gamma_k)} \frac{g(\gamma_k)}{G'(\gamma_k)} \frac{p}{p-\phi(\gamma_k)}. \tag{A 23}$$

For (A 21) and (A 22),

$$g(\gamma) = \cosh j\gamma, \quad G(\gamma) = \cosh n\gamma, \quad \phi(\gamma) = 2(\cosh\gamma - 1).$$

Thus $\gamma_0 = 0$, the roots of $G(\gamma) = 0$ are

$$\gamma_k = (2k+1)\, i\pi/2n \quad [0 < k \leqslant (n-1)] \tag{A 24}$$

(values of k greater than $(n-1)$ do not give distinct values of $\phi(\gamma)$) and

$$\left.\begin{aligned} \phi(\gamma_k) &= -4\sin^2[(2k+1)\,\pi/4n], \\ \phi'(\gamma_k) &= 2i\sin[(2k+1)\,\pi/4n], \\ G(\gamma_k) &= (-1)^k in. \end{aligned}\right\} \tag{A 25}$$

Hence from (A 23), the operator in (A 21) is

$$1 - \sum_{k=0}^{n-1} (-1)^k \frac{\cos[(2k+1)\, j\pi/2n]}{n\tan[(2k+1)\,\pi/4n]} \frac{p}{p+4\sin^2[(2k+1)\,\pi/4n]}. \tag{A 26}$$

As for the operational solution of the partial differential equations, the result of operating on the surface temperature with the terms in the sum in (A 26) can be expressed in terms of integrals by means of (A 8); but if the surface temperature is given by an analytical formula, it is often best to replace this by its operational representation and then interpret the resulting expression. For example, for

$$\begin{aligned} \theta_s(T) &= 0 \qquad (T<0) \\ &= \beta_1 T, \quad (T>0) \end{aligned}$$

has the operational representation $\theta_s(T) \equiv \beta_1 p^{-1}$,
so (A 21) and (A 26) give

$$\begin{aligned} \theta_j(T) &\equiv \beta_1 \left[p^{-1} - \sum_{k=0}^{n-1} (-1)^k \frac{\cos\{(2k+1)\, j\pi/2n\}}{n\tan\{(2k+1)\,\pi/4n\}} \frac{1}{p+4\sin^2\{(2k+1)\,\pi/4n\}} \right] \\ &\equiv \beta_1 \left[T - \frac{1}{4n} \sum_{k=0}^{n-1} (-1)^k \frac{\cos\{(2k+1)\, j\pi/2n\}}{\tan\{(2k+1)\,\pi/4n\} \sin^2\{(2k+1)\,\pi/4n\}} \left(1 - e^{-4T\sin^2[(2k+1)\,\pi/4n]}\right) \right]. \end{aligned} \tag{A 27}$$

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